First-Year Electromagnetism

Prof Laura Herz

www-herz.physics.ox.ac.uk/teaching.html

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

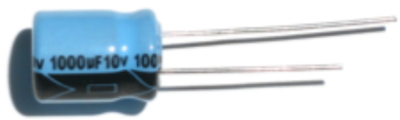
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Electromagnetism in everyday life

Electrostatics



Magnetostatics



Induction





Electromagnetic waves

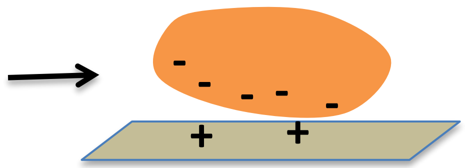


Electromagnetism in ancient history

Electrostatics

Ancient Greece: rubbing amber against fur allows it to attract other light substances such as dust or papyrus

+



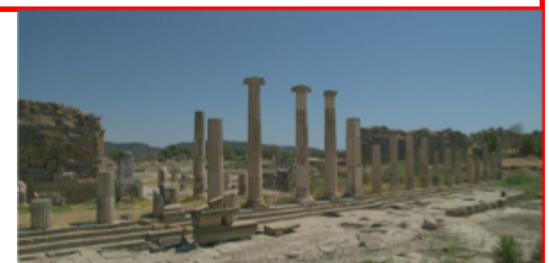
Greek word for "amber": *ἤλεκτρον* (*elektron*)

Magnetostatics

Magnesia (ancient Greek city in Ionia, today in Turkey):
Naturally occurring minerals were found to attract metal objects (first references ~600BC).

Crystals are referred to as: Iron ore, Lodestone, Magnetite, Fe_3O_4

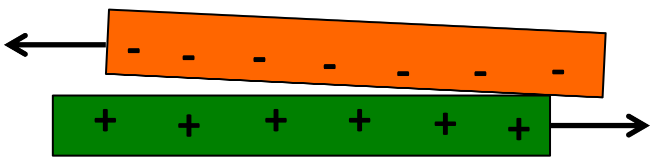
Use of Lodestone compass for navigation in medieval China



Electromagnetism in modern history

17^{th} century AD to mid 18^{th} century:

Dominated by "frictional electrostatics" – arising from "triboelectric effect":



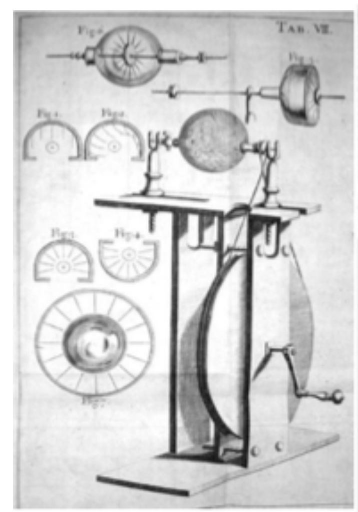
- When two different materials are brought into contact, charge flows to equalize their electro-chemical potentials, bonds form across surface
- Separating them may lead to charge remaining unequally distributed when bonds are broken
- Rubbing enhances effect through repeat contact

Focus on "electrostatic generators" – today's van de Graaff Generators:

Machines involved frictional passage of "positive" materials such as hair, silk, fur, leather against "negative" materials such as amber, sulfur

1660 Otto von Guericke, 1750ies, Hauksbee, Bose, Litzendorf, Wilson, Canton et al.

But: not creating high electric energy density



Electromagnetism in modern history

From late 18th century: *Rapid progress on both fundamental science and technology:*

- 1749: Benjamin Franklin invents lightning rod following experiments with kites.
- 1784: Charles-Augustin de Coulomb uses "torsion balance" to show that forces between two charged spheres vary with the square of the inverse distance between them.
- 1800: Alessandro Volta constructs the first electrochemical battery (zinc/copper/sulfuric acid) allowing high-density electrical energy storage
- 1821: André-Marie Ampère investigates attractive and repulsive forces between current-carrying wires
- 1831-55: Michael Faraday discovers electromagnetic induction by experimenting with two co-axial coils of wire, wound around the same bobbin.
- 1830ies: Heinrich Lenz shows that induced currents have a direction that opposes the motions that produce them
- 1831: first commercial telegraph line, from Paddington Station to West Drayton
- From 1850: construction of electromagnetic machines (Pixii, Varley, Siemens, et al.)
- 1861: overland telegraph line connects east and west coast of the United States
- 1864: James Clerk Maxwell introduces unified theory of electromagnetism, including a link to light waves
- 1887: Heinrich Hertz demonstrates the existence of electromagnetic waves in space
- Late 19th century: development of "wireless telegraphy" – radio!

Structure of the Course

1. Electrostatics

Charges create "electric fields" which represent the resulting force experienced by a small test charge.

$$\boxed{\oint_V \mathbf{E} \cdot \mathbf{dA} = \frac{Q_V}{\varepsilon_0}}$$

2. Magnetostatics

Electrical currents create "magnetic fields" which create forces on moving test charges. There are no magnetic monopoles.

$$\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot \mathbf{dl} = I$$

$$\boxed{\oint_V \mathbf{B} \cdot \mathbf{dA} = 0}$$

3. Induction

A time-varying magnetic flux through an area creates an electromotive force along the area's rim.

$$\boxed{\oint_{\partial A} \mathbf{E} \cdot \mathbf{dl} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathbf{B} \cdot \mathbf{dS}}$$

4. Electromagnetic waves

A time-varying electric flux through an area creates an magnetic field along the area's rim.

$$\boxed{\frac{1}{\mu_0} \oint_{\partial A} \mathbf{B} \cdot \mathbf{dl} = I + \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_A \mathbf{E} \cdot \mathbf{dS}}$$

Electromagnetic waves can propagate

Syllabus of the Course

1. Electrostatics

Coulomb's law. The electric field E and potential due to a point charge and systems of point charges, including the electric dipole. The couple and force on, and the energy of, a dipole in an external electric field. Energy of a system of point charges; energy stored in an electric field. Gauss' Law; the E field and potential due to surface and volume distributions of charge (including simple examples of the method of images), no field inside a closed conductor. Force on a conductor. The capacitance of parallel-plate, cylindrical and spherical capacitors, energy stored in capacitors.

2. Magnetostatics

The forces between wires carrying steady currents. The magnetic field B, Ampere's law, Gauss' Law ("no magnetic monopoles"), the Biot-Savart Law. The B field due to currents in a long straight wire, in a circular loop (on axis only) and in straight and toroidal solenoids. The magnetic dipole; its B field. The force and couple on, and the energy of, a dipole in an external B field. Energy stored in a B field. The force on a charged particle in E and B fields.

3. Induction

Electromagnetic induction, the laws of Faraday and Lenz. EMFs generated by an external, changing magnetic field threading a circuit and due to the motion of a circuit in an external magnetic field, the flux rule. Self and mutual inductance: calculation for simple circuits, energy stored in inductors. The transformer.

4. Electromagnetic waves

Charge conservation, Ampere's law applied to a charging capacitor, Maxwell's addition to Ampere's law ("displacement current"). Maxwell's equations for fields in a vacuum (rectangular coordinates only). Plane electromagnetic waves in empty space: their speed; the relationships between **E**, **B** and the direction of propagation.

Text Books

Introductory undergraduate textbooks on electromagnetism:

D. J. Griffiths, *Introduction to Electromagnetism*
Pearson, 4th edition, ISBN: 978 0 321 84781 2

I. S. Grant and W. R. Phillips, *Electromagnetism*
John Wiley, 2nd edition, ISBN: 978 0 471 92712 9

E. M. Purcell and D. J. Morin, *Electricity and Magnetism*
Pearson, 4th edition, ISBN: 978 1 107 01402 2

P. Lorrain, D. R. Corson and F. Lorrain, Fundamentals of Electromagnetic Phenomena
Freeman, ISBN: 978 0 716 73568 7

Also of interest:

W. J. Duffin, *Electricity and Magnetism*
Duffin Publishing (out of print)

Feynman, Leighton, Sands, *The Feynman Lectures on Physics, Vol II*
ISBN: 978 0 465 02382 0

W. G. Rees, *Physics by Example*
Cambridge University Press, ISBN: 978 0 521 44975 5

1. Electrostatics

1.1. Introduction: Properties of charge; Coulomb's law

1.2. The Principle of Superposition

1.3. The Electric Field and Electrostatic Potential

1.4. Assemblies of discrete charges; multipoles

1.5. Continuous charge distributions

Problem Set 1

1.6. Gauss' law

1.7. Poisson and Laplace equations

1.8. The Method of Image Charges

Problem Set 2

1.9. Capacitance and Energy of the Electric Field

Problem Set 3

1.1. Properties of Charge - Summary

- Both positive and negative charge exists (triboelectric experiments showed electrostatic attraction and repulsion)

- Charge is quantized (Millikan experiment, 1913): e=1.602×10^{-19} As

- Coulomb's law (1785): the force between two point charges varies with the square of their inverse distance:

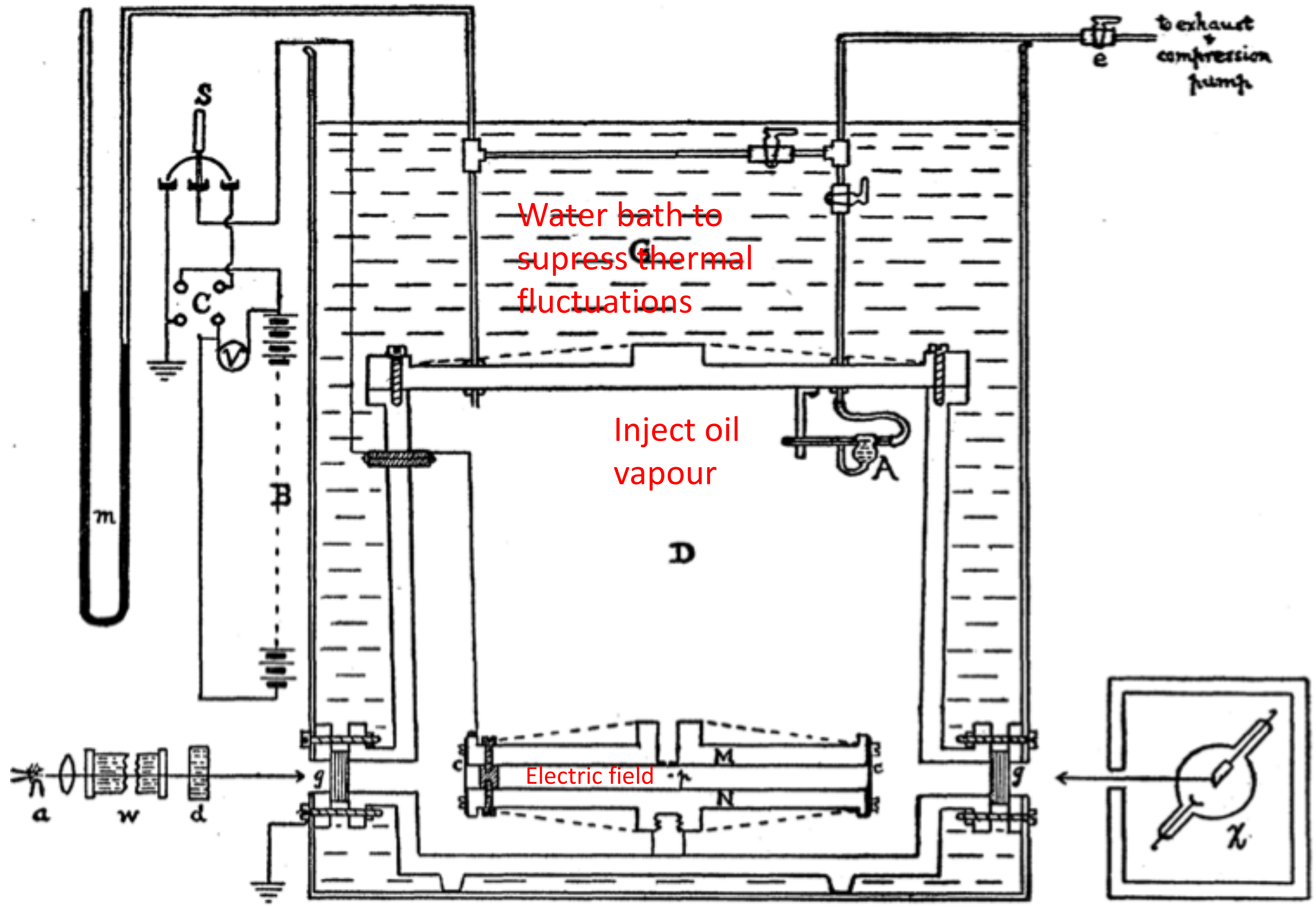
$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

- Superposition: The force between two point charges varies linearly with the amount of each charge, hence the forces resulting from individual charges superimpose in an assembly of charges: $\mathbf{F} = \Sigma\, \mathbf{F}_i$

1.1. Properties of Charge – Millikan Experiment

Millikan experiment, Phys. Rev. 2, 109–143 (1913):

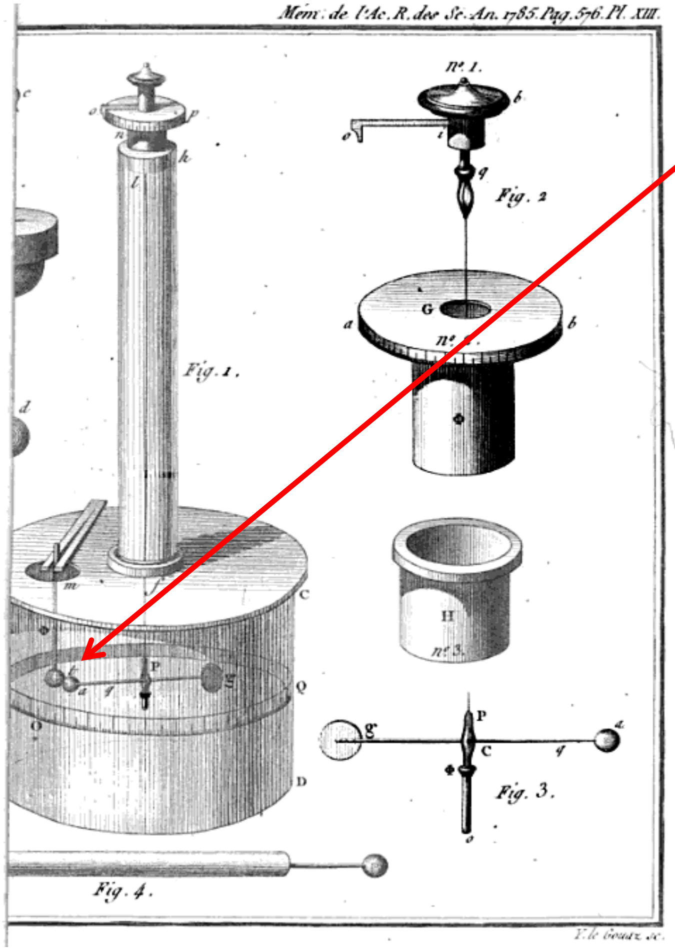
Observe oil droplet motion in homogeneous electric force field



Ionize oil droplets with X-rays

1.1. Properties of Charge – Coulomb's law

Coulomb's Torsion Balance experiment, *Histoire de l'Academie Royale des Science*, p. 569-577 (1785):



Measure force between two charged spheres through torsion force on wire:

He found: $$F \propto \frac{1}{r^2}$$

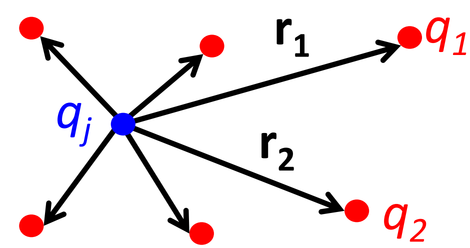
Coulomb's law: $$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$$

$$\varepsilon_0 = 8.854 \times 10^{-12}\frac{As}{Vm}$$

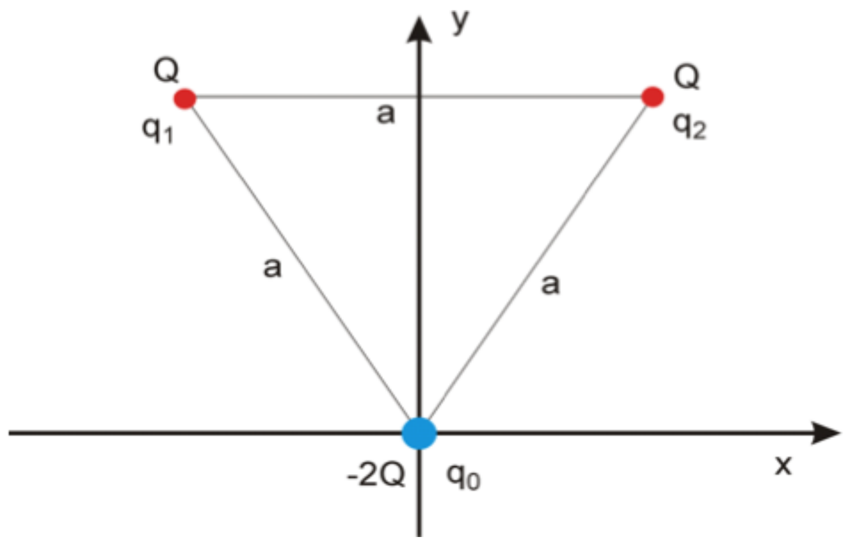
1.2. The Principle of Superposition

The force on charge q_j originating from all other charges q_i is given by:

$$\mathbf{F_j} = \frac{q_j}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_i}{r_{ji}^2}\hat{\mathbf{r_{ji}}}$$



Example 1: force on charge -2Q resulting from two charges Q in the corners of a triangle:



$$\mathbf{F} = \frac{\sqrt{3}Q^2}{2\pi\varepsilon_0 a^2}\hat{\mathbf{e_y}}$$

1.3. The Electric Field and Electrostatic Potential

The electric field **E** at a point **r**, generated by a distribution of charges q_i, is equal to the force **F** per unit charge q that a small test charge q would experience if it was placed at **r**:

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}(\mathbf{r})}{q}$$

The electric potential V at a point **r** is the energy W required per unit charge q to move a small test charge q from a reference point to **r**. For a system of charges:

$$V(\mathbf{r}) = \frac{W(\mathbf{r})}{q}$$

The electric field and potential are related through:

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot \mathbf{dr}' \quad \longleftrightarrow \quad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

1.4. Assemblies of Discrete Charge Systems

The Electric field **E** and Potential *V* of a distribution of point charges q_i placed at positions $\mathbf{r}_i$ are:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{(\mathbf{r}-\mathbf{r}_i)^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|}$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}$$

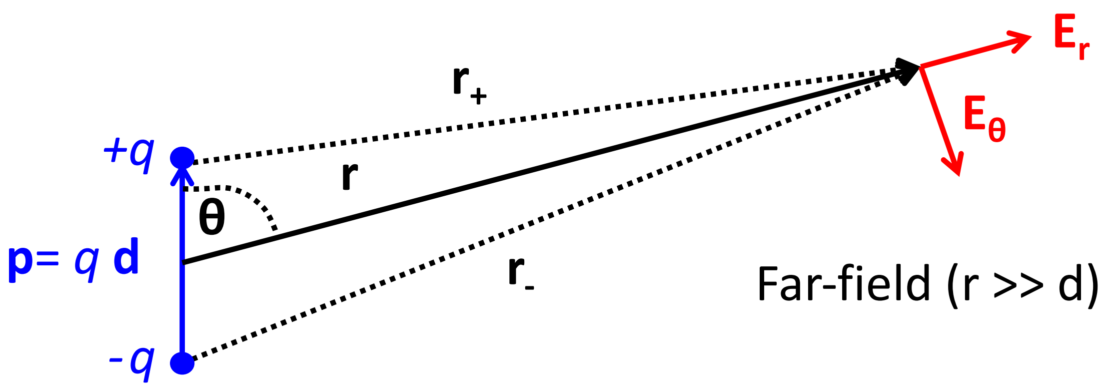
The energy *U* required to assemble a system of point charges q_i by bringing them to positions $\mathbf{r}_i$ from infinity is given by:

$$U = \frac{1}{8\pi\varepsilon_0}\sum_i q_i \sum_{i\neq j}\frac{q_j}{r_{ji}} = \frac{1}{2}\sum_i q_i V_i$$

where V_i is the potential experienced by q_i at $\mathbf{r}_i$ from all *other* charges q_j.

1.4. Assemblies of Discrete Charge Systems

Example 2: Electric Dipole

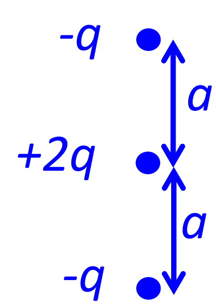


Far-field (r >> d) potential: $$V = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0\, r^3}$$

Far-field (r >> d) electric field: $$\mathbf{E}_r = \frac{2\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0\, r^4} \quad \mathbf{E}_\theta = \frac{qd\sin\theta}{4\pi\varepsilon_0\, r^3} \quad \mathbf{E}_\phi = 0$$

Energy of dipole in external electric field $\mathbf{E}_{ext}$: $$W = -\mathbf{E}_{\rm ext}\cdot\mathbf{p}$$

Example 3: Electric Quadrupole:

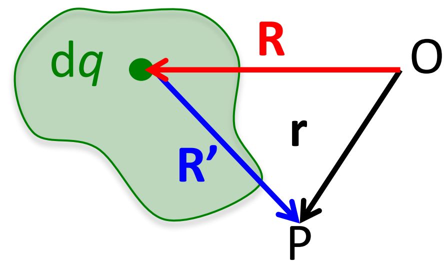


$$V = \frac{qa^2}{4\pi\varepsilon_0\, r^3}(1-3\cos^2\theta)$$

1.5. Continuous Charge Distributions

Continuous charge distribution:

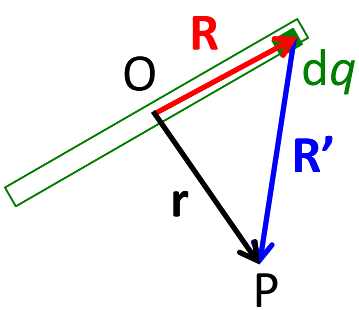
$$\sum V_i \rightarrow \int \mathrm{d}V$$



$$V = \int \frac{\mathrm{d}q}{4\pi\varepsilon_0 \left|\mathbf{r} - \mathbf{R}\right|}$$

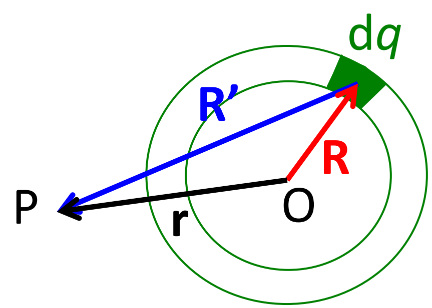
Line charge:

$\mathrm{d}q = \lambda \, \mathrm{d}l$



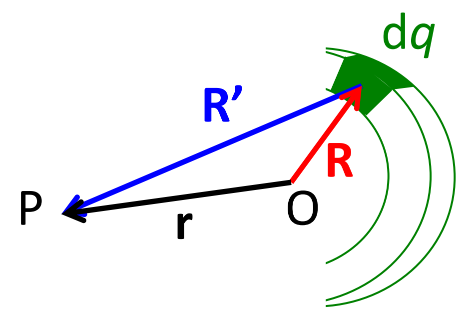
Surface charge:

$\mathrm{d}q = \sigma \, \mathrm{d}A$



Volume charge:

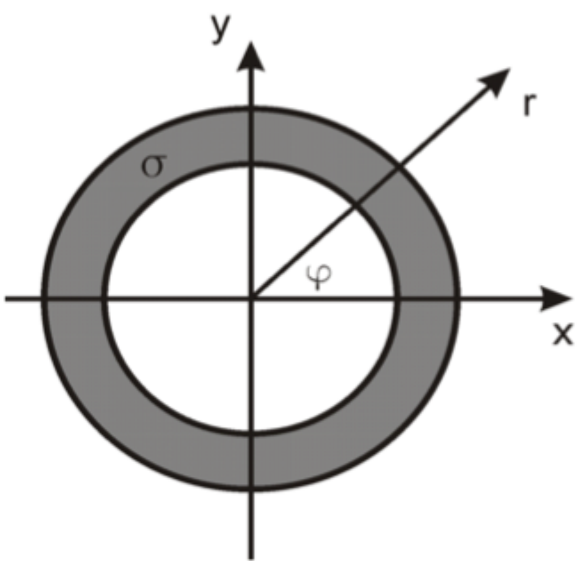
$\mathrm{d}q = \rho \, \mathrm{d}V$

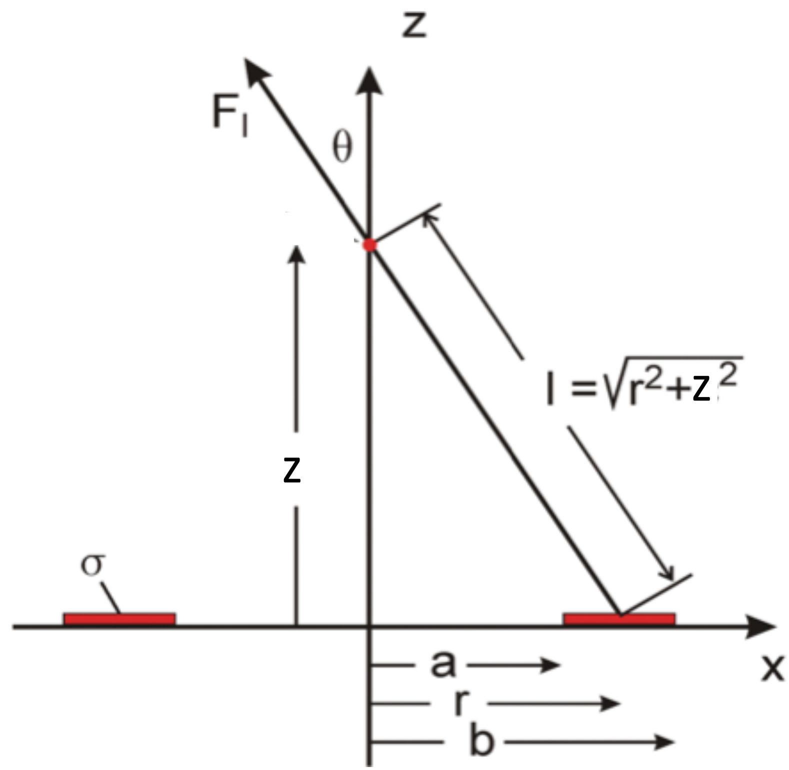


Choose a convenient origin O suiting the geometry of the charge distribution!

1.5. Continuous Charge Distributions

Example 4: Uniformly charged ring.



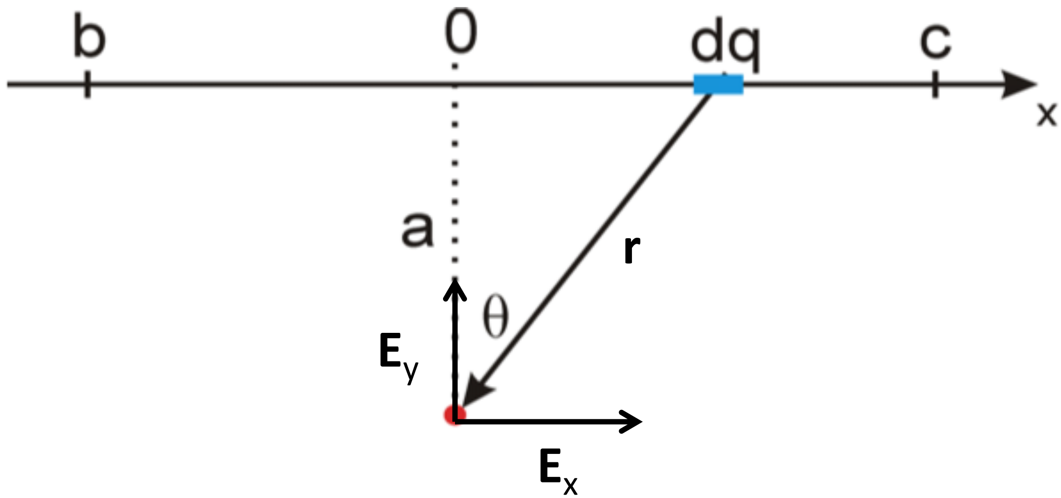


$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2+z^2} - \sqrt{a^2+z^2}\right]$$

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{\sqrt{a^2+z^2}} - \frac{z}{\sqrt{b^2+z^2}}\right]\hat{\mathbf{e}}_z$$

1.5. Continuous Charge Distributions

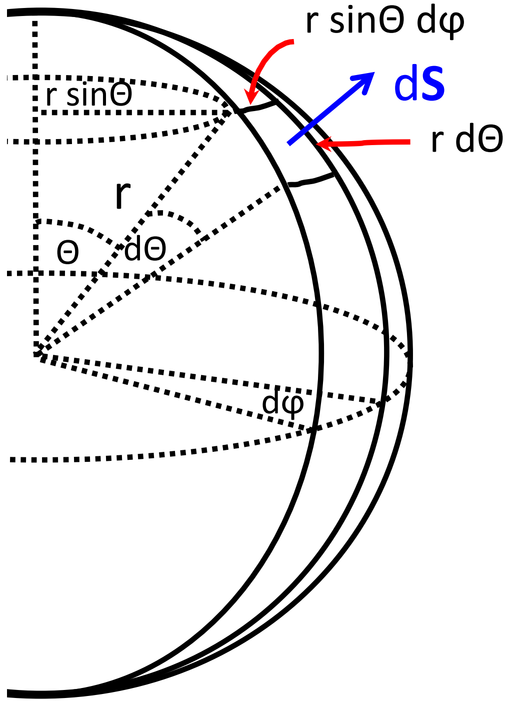
Example 5: Uniformly charged rod.



$$\mathbf{E}_x = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{a^2+c^2}} - \frac{1}{\sqrt{a^2+b^2}}\right]$$

$$\mathbf{E}_y = \frac{-\lambda}{4\pi\varepsilon_0 a}\left[\frac{c}{\sqrt{a^2+c^2}} + \frac{b}{\sqrt{a^2+b^2}}\right]$$

1.6. Gauss's Law



Area and Solid angle elements:

$dS = r^2 \sin\Theta \, d\Theta \, d\varphi = r^2 \, d\Omega$

Calculate electric field flux dΦ through area dS for a point charge q_i a distance r away from dS:

$$d\Phi = \mathbf{E}_i \cdot d\mathbf{S} = \frac{q_i}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \cdot \hat{\mathbf{r}} \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= \frac{q_i}{4\pi\varepsilon_0} \underbrace{\sin\theta \, d\theta \, d\phi}_{d\Omega}$$

Independent of r !

Integrate over a closed surface:

$$\oint \mathbf{E}_i \cdot d\mathbf{S} = \frac{q_i}{\varepsilon_0} \frac{\oint d\Omega}{4\pi} = \begin{cases} \dfrac{q_i}{\varepsilon_0} & \text{if } q_i \text{ is enclosed} \\ 0 & \text{if } q_i \text{ is not enclosed} \end{cases}$$

Principle of superposition →

Gauss' Law;

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}$$

E-field on surface

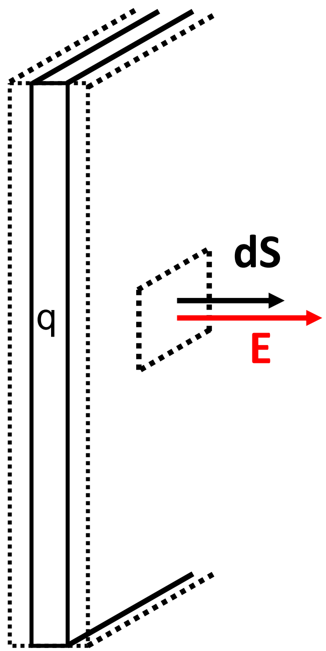
total charge enclosed

1.6. Gauss's Law: Applications

Using Gauss's law to find the electric field of a charge distribution:

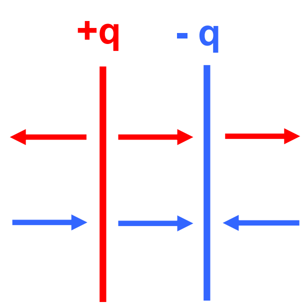
Need to:
- Find a surface on which **E•dS** is the same at any surface point
- Be careful with edge effects!

Example 6: Uniformly charged, "infinite" plate of area A.



$$\oint \mathbf{E}\cdot \mathrm{d}\mathbf{S} = E \times \underbrace{2A}_{\text{both sides}} = \frac{q}{\varepsilon_0} \longrightarrow E = \frac{1}{2\varepsilon_0}\underbrace{\frac{q}{A}}_{\sigma}$$

For a pair of oppositely charged plates (plate capacitor):



Principle of superposition:

Between the plates: $E = \frac{\sigma}{2\varepsilon_0} - \frac{-\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$

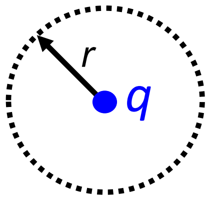
Around both plates: $E = \frac{\sigma}{2\varepsilon_0} + \frac{-\sigma}{2\varepsilon_0} = 0$

1.6. Gauss's Law: Applications

Example 7: Spherically symmetric charge distributions.

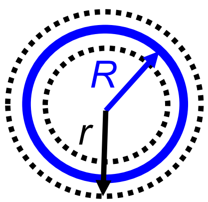
$$\oint_{\delta V} \mathbf{E}\cdot d\mathbf{S} = E_r \times 4\pi r^2 = \frac{1}{\varepsilon_0}\int_V \rho\, dV \longrightarrow E_r = \frac{1}{4\pi\varepsilon_0 r^2}\int_V \rho\, dV$$

(i) point charge q:



$$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$ for any r

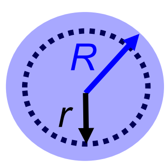
(ii) hollow sphere with q spread evenly across surface:



For 0 < r < R (inside sphere): $E_r = 0$

For R < r (outside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

(iii) Sphere carrying uniform volume charge ρ:

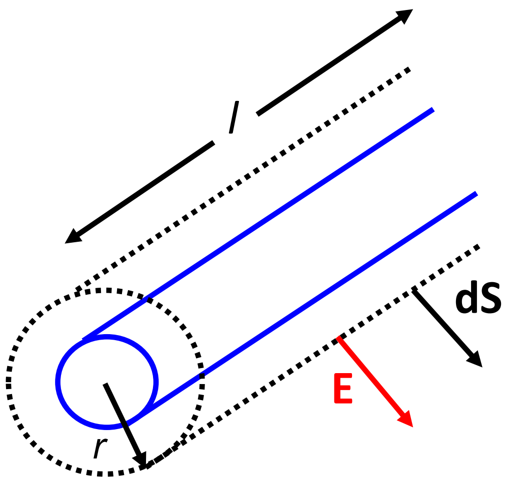


For 0 < r < R (inside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 R^2}\frac{r}{R}$$

For R < r (outside sphere): $$E_r = \frac{q}{4\pi\varepsilon_0 r^2}$$

1.6. Gauss's Law: Applications

Example 8: Long, uniformly charged rod.



$$\oint_{\delta V} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = E_r \times 2\pi\, r \times l = \frac{q}{\varepsilon_0}$$

line charge λ

$$E_r = \frac{q/l}{2\pi\varepsilon_0\, r}$$

for R < r (outside rod)

1.6. Gauss's Law: Conductors

Example 9: Electric fields and charge distributions inside a conductor.

Inside a conductor, one or more electrons per atom are free to move throughout the material (copper, gold, and other metals). As a result:

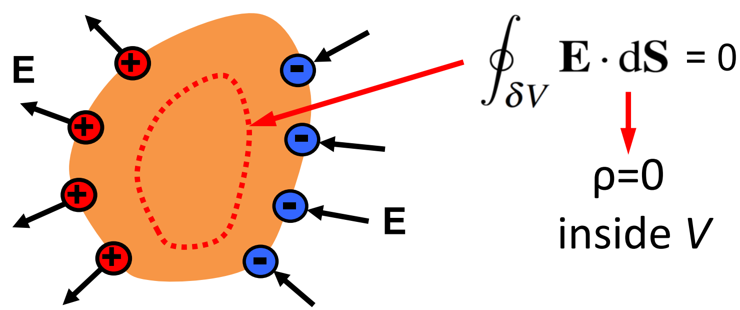
(i) **E**=0 inside a conductor (free charge moves to surface until the internal electric field is cancelled).

(ii) ρ=0 inside a conductor (from Gauss' law: **E**=0 hence ρ=0).

(iii) Therefore any net charge resides on the surface.

(iv) A conductor is an equipotential (since **E**=0, V($\mathbf{r}_1$)=V($\mathbf{r}_2$)).

(v) At the surface of a conductor, **E** is perpendicular to the surface (otherwise charges will flow until the tangential component becomes zero when equilibrium is reached).



1.7. Poisson and Laplace Equations

Gauss' law: $$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$ Definition of Potential: $$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson equation}}$$

In regions where ρ=0: $$\nabla^2 V = 0$$ Laplace equation

Uniqueness Theorem:

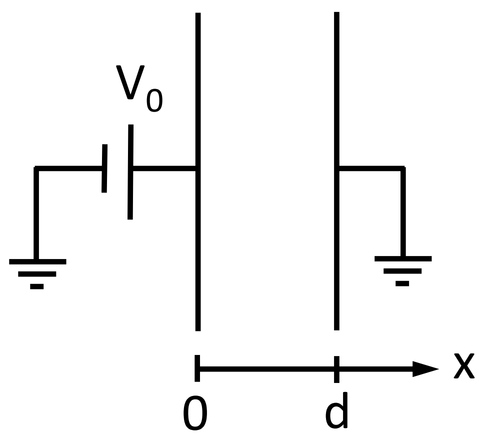
The potential V inside a volume is *uniquely* determined, if the following are specified:

(i) The charge density throughout the region

(ii) The value of V on all boundaries

1.7. Laplace Equations: Solutions for Special Cases

Example 10: Solutions to Laplace's equation for a parallel-plate capacitor.



Symmetry suggests use of cartesian coordinates:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \underbrace{\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}}_{0} = 0 \longrightarrow V(x) = C_1 x + C_2$$

Boundary conditions:

$V(0) = V_0$ and $V(d) = 0$

$$\longrightarrow \quad V(x) = V_0(1 - x/d)$$

and $$\mathbf{E} = -\nabla V = \frac{V_0}{d}\hat{\mathbf{e}}_x$$

1.7. Laplace Equations: Solutions for Special Cases

Example 11: General solutions to Laplace's equation for charge distributions with azimuthal symmetry.

$$\frac{\partial V}{\partial \phi} = 0 \longrightarrow \nabla^2 V = \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0$$

Separation of variables yields the general solutions:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

where A_l, B_l are constants determined by boundary conditions and P_l are Legendre Polynomials in cos θ, i.e.:

$$V(r,\theta) = A_0 + \frac{B_0}{r} + A_1 r\cos\theta + \frac{B_1}{r^2}\cos\theta$$
$$+ A_2 r^2 \frac{1}{2}(3\cos^2\theta - 1) + \frac{B_2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \cdots$$

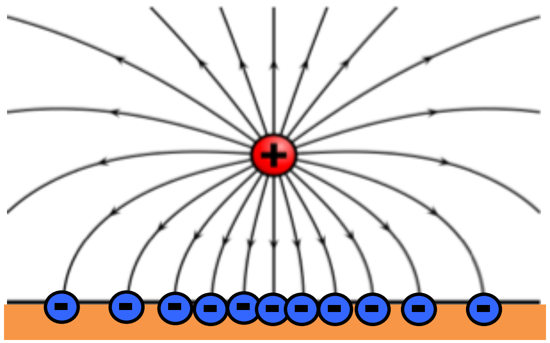
1.8. The Method of Image Charges

The Method of Image Charges:

- Useful for calculating potentials created by charges placed in the vicinity of metal conductors
- Replace metal elements with imaginary charges (“image charge”) which replicate the boundary conditions of the problem on a surface.
- The Uniqueness Theorem guarantees that within the region bounded by the surface, the potential calculated for the “imagined” charge distribution is identical to that of the “real” situation.
- If a suitable replacement “image charge distribution” is chosen, the calculation of the potential becomes mathematically much simpler.

1.8. The Method of Image Charges

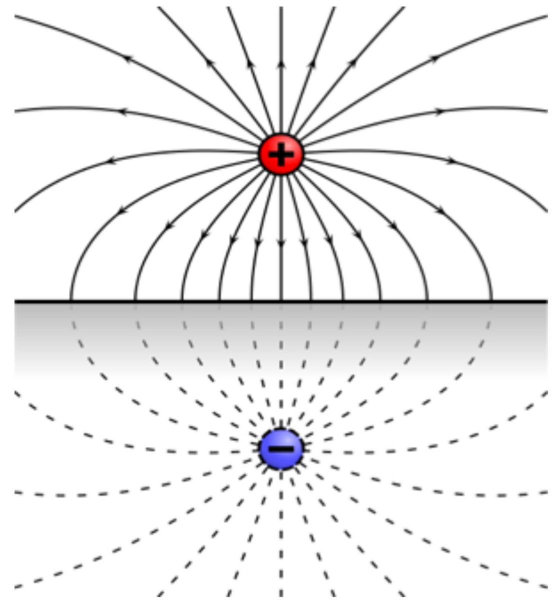
Point charge a distance d above a grounded metal plate:



equivalent

scenarios

Two opposite point charges a distance 2d apart:



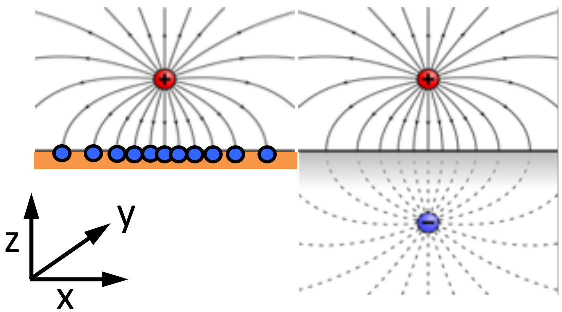
Boundary conditions:

1. Metal surface is equipotential (V=0) – hence no tangential E-field component.
2. Far from the plate and the point charge, the potential must drop to zero.

The two assemblies share the same charge distribution and boundary conditions for the upper volume half. The Uniqueness Theorem states that the potential in those regions must therefore be identical!

1.8. The Method of Image Charges

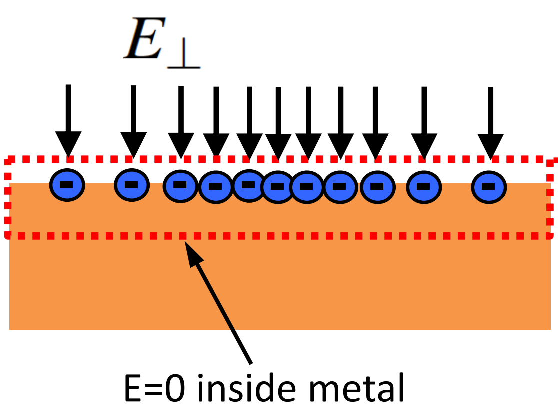
Point charge *q* a distance *d* above a grounded metal plate:



Potential (derived from image charge scenario):

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Induced surface charge:



Gauss' law

$$\sigma_{\text{ind}} = \varepsilon_0 E_\perp = i\varepsilon_0 \frac{\partial V}{\partial n}\Big|_{\text{surface}}$$

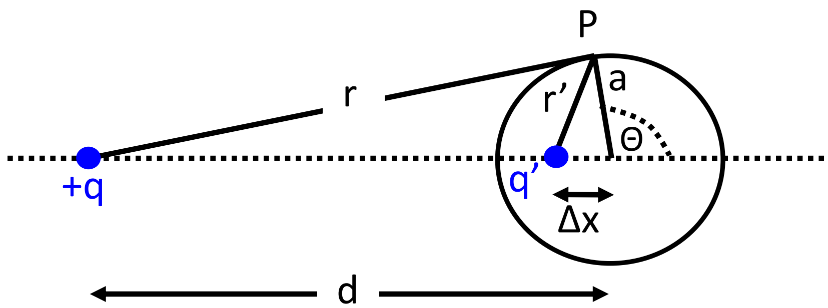
$$q_{\text{ind}} = \int_0^\infty \int_0^\infty \sigma_{\text{ind}}\, \mathrm{d}x\, \mathrm{d}y = -q$$

Force between charge and plate:

$$\mathbf{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{\mathbf{e}}_z$$

1.8. The Method of Image Charges

Example 12: Point charge outside a grounded metal sphere of radius *a*.



Potential on metal sphere is constant.

Try to find point charge q' which replaces the metal sphere and results in V=0 for points on the sphere surface.

Potential at points P arising from q and q' alone: $V = -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$

with $r^2 = a^2 + d^2 + 2ad\cos\theta$ and $r'^2 = a^2 + \Delta x^2 + 2a\Delta x\cos\theta$

Taking the condition V=0 for all points P on the sphere one finds:

$$\Delta x = \frac{a^2}{d}$$

$$q' = -q\frac{a}{d}$$

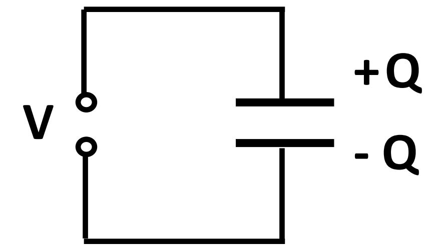
V for all points outside the sphere:

$$V = -\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{(x-d)^2+y^2+z^2}} + \frac{-q\frac{a}{d}}{\sqrt{(x-\frac{a^2}{d^2})^2+y^2+z^2}}\right)$$

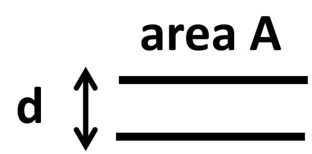
1.9. Capacitance

Capacitance: Storage of energy through separation of two oppositely poled charge accumulations

$$\text{Capacitance } C = \frac{\text{charge } Q}{\text{voltage } V \text{ applied}}$$

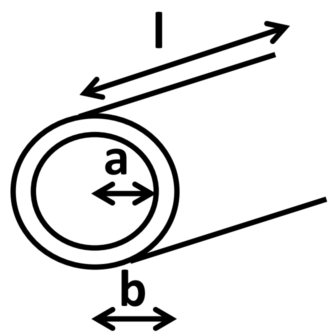


(i) parallel-plate capacitor:



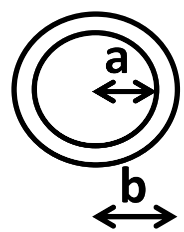
$$C = \varepsilon_0 \frac{A}{d}$$

(ii) cylindrical capacitor:



$$\frac{C}{l} = \frac{2\pi\varepsilon_0}{\ln(b/a)}$$

(iii) spherical capacitor:



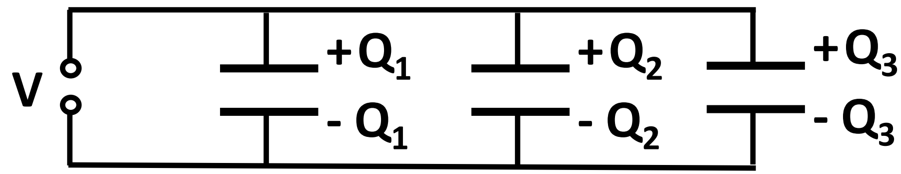
$$C = 4\pi\varepsilon_0 \frac{a\,b}{b-a}$$

1.9. Capacitance

Capacitance networks:

(i) Capacitors in parallel:

$$C = C_1 + C_2 + C_3$$

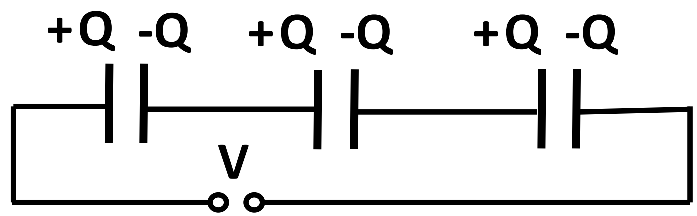


V=V1=V2=V3

Q= ΣQi

(ii) Capacitors in series:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$



Q=Q1=Q2=Q3

V= ΣVi

Energy stored in a capacitor:

Capacitor initially uncharged -> add a small amount of charge
Further charge has to be brought in against the potential created by the existing charge:

$$W = \int_0^{V_0} V(q)dq = \int_0^{V_0} VCdV$$

Energy W stored in capacitor:

$$W = \frac{1}{2}CV_0^2 = \frac{1}{2}\frac{Q_0^2}{C} = \frac{1}{2}Q_0V_0$$

1.9. Energy of the Electric Field

Energy W associated with a charge distribution and its electric field:

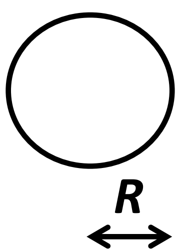
(i) for a parallel-plate capacitor:

$$W = \frac{1}{2}\varepsilon_0 E^2 \underbrace{A\,d}_{\text{volume in between plates}}$$

(ii) for a general continuous charge distribution ρ:

$$W = \frac{1}{2}\int_v \rho\, V\, dv \longrightarrow \boxed{W = \frac{\varepsilon_0}{2}\int E^2 dv \quad \text{integrating over all space}}$$

(iii) for a hollow sphere of radius R carrying charge Q:



$$W = \frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0 R}$$

2. Magnetostatics

2.1. Introduction: Origins of Magnetism

2.2. Forces on Current-Carrying Wires in Magnetic Fields

2.3. Current Density and the Continuity Equation

2.4. The Biot-Savart Law (B-fields of Wires, Solenoids, etc.)

2.5. Magnetic Dipoles

2.6. Ampere’s Law & Gauss’ Law of Magnetostatics

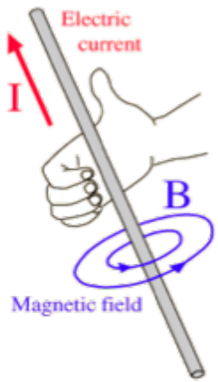
2.7. Magnetic Scalar and Vector Potentials

Problem Set 3

2.1. Introduction: Origins of Magnetism

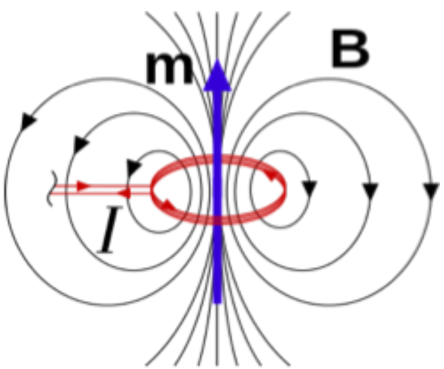
Minerals found in ancient Greek city Magnesia ("Magnetite", Fe_3O_4) attract small metal objects.

Materials containing certain atoms such as Iron (Fe), Cobalt (Co), Nickel (Ni) can exhibit "permanent" magnetic dipoles.



Forces exist between pairs of current-carrying wires (attractive for current flowing in the same, repulsive for current flowing in opposite directions).

An electric current through a wire creates magnetic fields whose field lines loop around the wire.



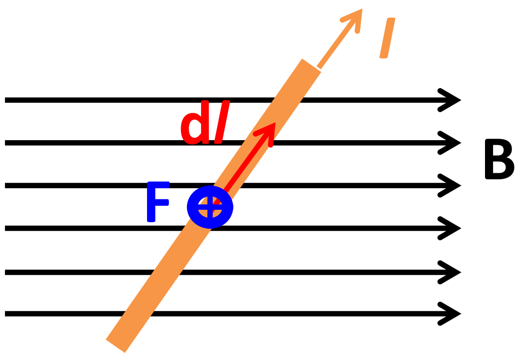
Magnetic field lines form closed loops or disappear at infinity. They do not originate from "magnetic monopoles".

The fundamental generators of magnetic fields are dipoles that may result from electrical current loops or inherent material properties such as aligned angular momenta of charged particles.

2.2. Forces on current-carrying wires in magnetic fields

Define: *Magnetic Flux Density* **B**, *Magnetic Field* **H**, and **B**=μ_0**H** for non-magnetic materials

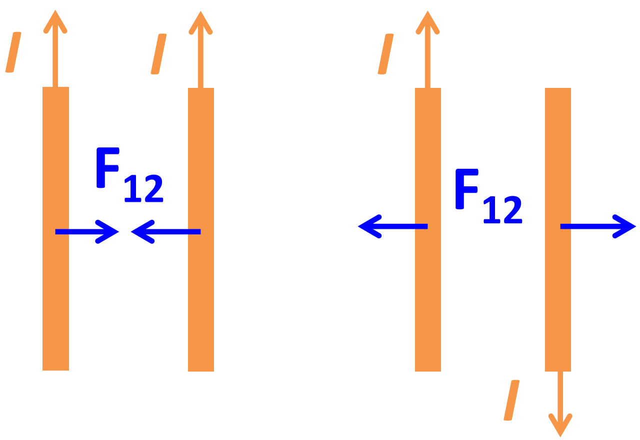
A current-carrying wire in a magnetic flux density **B** experiences a force **F** where:



$$\mathrm{d}F = I\,\mathrm{d}l\,\sin\theta\,B$$

$$\mathrm{d}\mathbf{F} = I\,\mathrm{d}\mathbf{l}\times\mathbf{B}$$

Two wires attract (repel) one another if they carry electrical current in the same (opposite) directions.



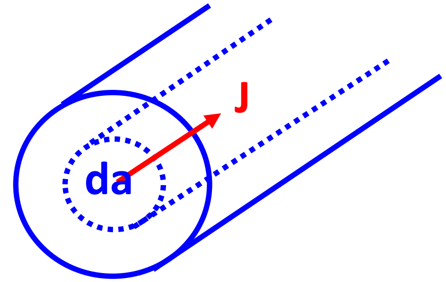
$$\mathbf{F}_{12} = \oint_{L_1}\oint_{L_2} \mu_0\, I_1\, I_2\, \frac{\mathrm{d}\mathbf{l_2}\times(\mathrm{d}\mathbf{l_1}\times\mathbf{r_{12}})}{4\pi\, r_{12}^3}$$

(from Biot-Savart's law – Example 13)

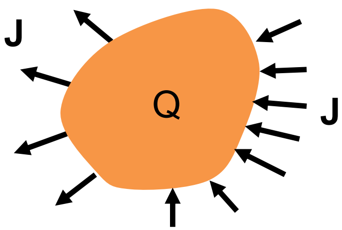
2.3. Current Density and the Continuity Equation

Define current density **J**:

$$\mathbf{J} = \frac{\mathrm{d}\mathbf{I}}{\mathrm{d}a_\perp}$$



For a closed volume, the net current entering must be equal to the rate in change of charge inside the volume (charge conservation):



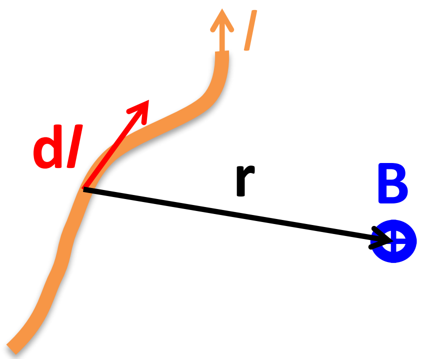
Continuity Equation:

$$\boxed{\oint_S \mathbf{J}\cdot\mathrm{d}\mathbf{a} = I = -\frac{\partial Q}{\partial t} \longleftrightarrow \nabla\cdot\mathbf{J} = -\frac{\partial \rho}{\partial t}}$$

In the limit of electro/magneto-statics:

$$\underbrace{\frac{\partial \mathbf{J}}{\partial t} = 0 \quad \text{steady currents}}_{\text{constant B-fields}} \qquad \underbrace{\frac{\partial \rho}{\partial t} = 0 \quad \text{stationary charges}}_{\text{constant E-fields}} \xrightarrow{\text{C E}} \nabla\cdot\mathbf{J} = 0$$

2.4. The Biot-Savart Law

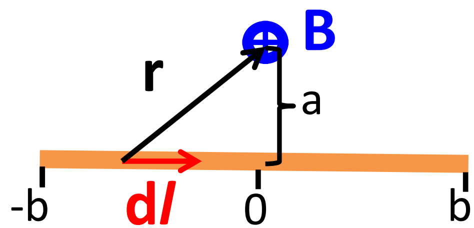


The magnetic flux density **B** created by a current loop is given by:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int I\,\frac{\mathrm{d}l \times \hat{\mathbf{r}}}{r^2} \quad \overset{\text{or}}{\longleftrightarrow} \quad \mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}\times\mathbf{r}}{r^3}\,\mathrm{d}^3 r$$

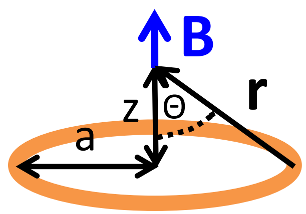
Biot-Savart Law

Example 14: straight wire.



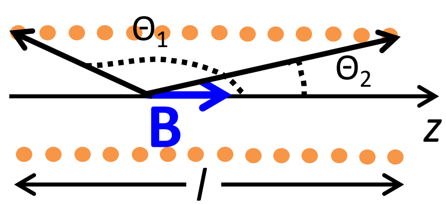
$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\frac{1}{(a^2/b^2+1)^{1/2}}$$

Example 15: circular loop.



$$\mathbf{B} = \frac{\mu_0 I a^2}{2\sqrt{z^2+a^2}^{\,3}} = \frac{\mu_0 I}{2a}\sin^3\theta$$

Example 16: solenoid.

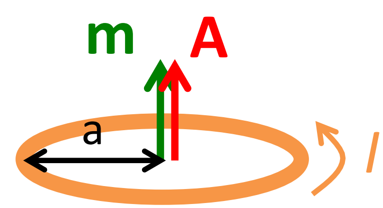


$$\mathbf{B} = \frac{\mu_0 I N}{2l}(\cos\theta_2 - \cos\theta_1)$$

2.5. Magnetic Dipoles

Magnetic dipole moment *m* of a current loop = current × area of the loop:

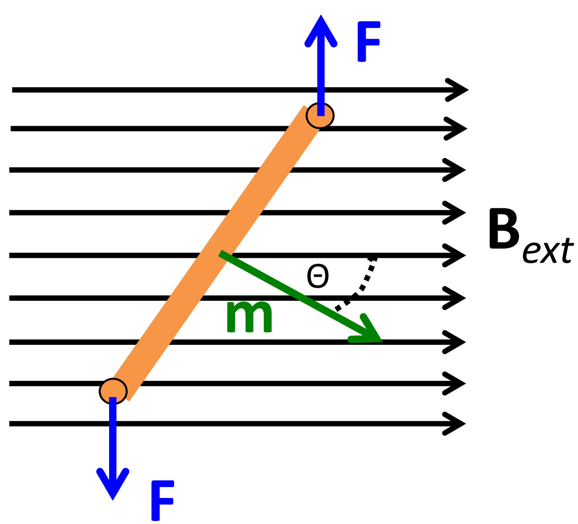
$$\mathbf{m} = I\,\mathbf{A}$$



$A=\pi a^2$

Magnetic flux density of a magnetic dipole:

$$B_r = \mu_0 \frac{2m\cos\theta}{4\pi r^3} \qquad B_\theta = \mu_0 \frac{m\sin\theta}{4\pi r^3} \qquad B_\phi = 0$$



Torque on a magnetic dipole in an external magnetic field $\mathbf{B}_{\text{ext}}$:

$$\mathbf{T} = I\,\mathbf{A} \times \mathbf{B}_{ext} = \mathbf{m} \times \mathbf{B}_{ext}$$

Energy of a magnetic dipole in an external magnetic flux density $\mathbf{B}_{\text{ext}}$:

$$W = -m B_{ext}\cos\theta = -\mathbf{m}\cdot\mathbf{B}_{ext}$$

2.6. Ampere's law and Gauss' law of magnetostatics

Ampere's law:

$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I$$

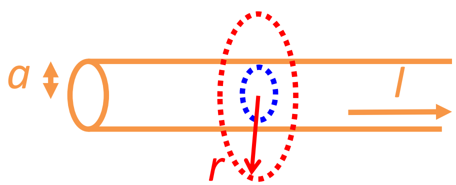
Electric currents generate magnetic fields whose field lines form closed loops.

"Gauss' law of Magnetism":

$$\oint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

There are no magnetic monopoles.

Example 17: infinite straight wire.



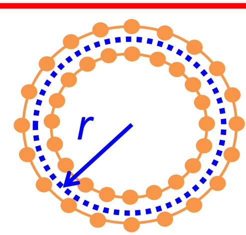
outside: $r > a$

$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,2\pi r = \mu_0 I \rightarrow B = \frac{\mu_0 I}{2\pi r}$$

inside: $0 < r < a$

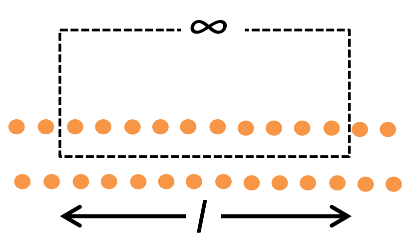
$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,2\pi r = \mu_0 I \frac{\pi r^2}{\pi a^2} \rightarrow B = \frac{\mu_0 I}{2\pi a^2} r$$

Example 18:
toroidal coil.



$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,2\pi r = \mu_0 N I \rightarrow B = \frac{\mu_0 N I}{2\pi r}$$

Example 19:
infinite solenoid.



z

$$\oint_{\partial A} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = B\,l = \mu_0 N I \rightarrow B = \mu_0 I \frac{N}{l}$$

2.7. Magnetic Scalar and Vector Potentials

Magnetic vector potential A defined through: $\mathbf{B} = \nabla \times \mathbf{A}$

Such **A** always exists because: $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

Inserting into Ampere's law:

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$$

There is a certain degree of freedom in which **A** to choose – set: $\nabla \cdot \mathbf{A} = 0$

Poisson equations for magnetostatics:
(one for each **J** & **A** coordinate)

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$$

Magnetic scalar potential V_m:

$$\mathbf{B} = -\mu_0 \nabla V_m \longleftrightarrow V_m = -\frac{1}{\mu_0} \int_A^B \mathbf{B} \cdot d\mathbf{l}$$

Caution: V_m is *pathway-dependent and not single-valued* because $\nabla \times \mathbf{B} \neq 0$.

But V_m can be used with care in simply-connected, current-free regions.

Being a scalar, V_m is mathematically easier to use than the vector potential.

Summary for Electro- and Magnetostatics

Electrostatics

Coulomb's law: $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'$$

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \leftrightarrow \oint_V \mathbf{E}\cdot\mathbf{dA} = \frac{Q_V}{\varepsilon_0}$$

Maxwell 1: Gauss' law. Charge generates an electric field. Electric field lines begin and end on charge.

$$\nabla\times\mathbf{E} = 0 \leftrightarrow \oint_{\partial A} \mathbf{E}\cdot\mathbf{dl} = 0$$

Maxwell 3: There is a well-defined electric scalar potential V, with: $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$

Magnetostatics

Biot-Savart law: $$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{(\mathbf{r}-\mathbf{r}')^2}\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\mathrm{d}^3r'$$

$\nabla\cdot\mathbf{J} = 0$

$$\nabla\cdot\mathbf{B} = 0 \leftrightarrow \oint_V \mathbf{B}\cdot\mathbf{dA} = 0$$

Maxwell 2: There are no magnetic monopoles. Magnetic field lines form closed loops.

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} \leftrightarrow \oint_{\partial A} \mathbf{B}\cdot\mathbf{dl} = \mu_0 I$$

Maxwell 4: Electric currents generate magnetic fields.

3. Electromagnetic Induction

3.1. Introduction: Electromagnetic Induction

3.2. The Lorentz Force

3.3. Faraday's and Lenz's Laws of Induction

3.4. Self-Inductance and Mutual Inductance

3.5. The Transformer

3.6. Energy of the Magnetic Field

3.7. Charged Particles in E- and B-Fields

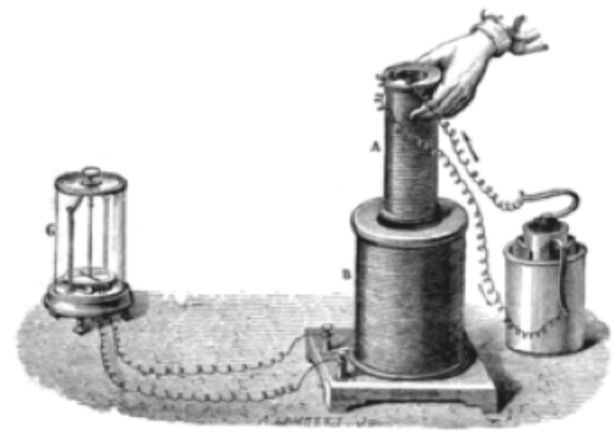
Problem Set 4

Problem Set 5

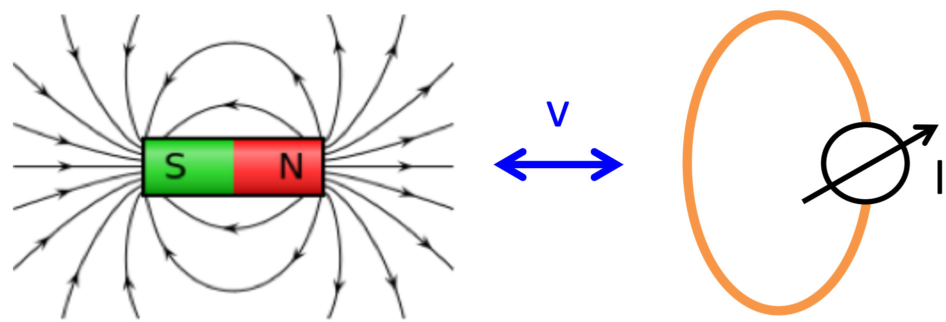
3.1. Introduction to Electromagnetic Induction

1831: Michael Faraday carries out a series of experiments and observes:

Two coils are arranged in a way so that the magnetic flux density of coil A penetrates through coil B. He finds that if the B-field in one coil is changing, this induces an electrical current in coil B.



Moving a bar magnet through a circuit element (wire loop) generates a current in the circuit. Moving instead the circuit w.r.t. the bar magnet, gives the same result.



→ A change with time in the magnetic flux density through a circuit causes an "electromotive force" that moves charges along the circuit.

3.2. The Lorentz Force

Lorentz Force on a point charge moving in a B-field:

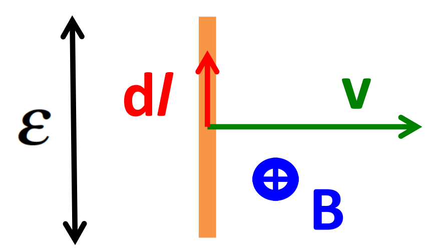
Any point charge q moving with velocity **v** in a magnetic flux density **B** experiences a Lorentz Force perpendicular to both **v** and **B**, with:

$$\mathbf{F}_{\mathrm{L}} = q\,\mathbf{v} \times \mathbf{B}$$

Electromotance resulting from forces on charges in a conductor moving with respect to a B-field:

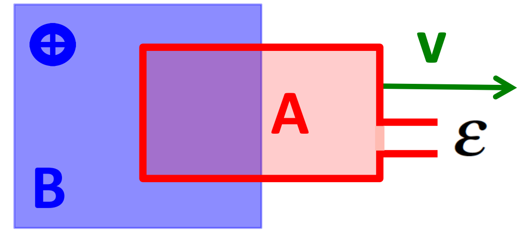
A Lorentz force on a charge in a circuit element **dl** moving with velocity **v** with respect to a magnetic flux density **B** causes an electromotance $\mathcal{E}$ in the circuit, with:

$$\mathcal{E} = \int_L (\mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l}$$



3.3. Faraday's and Lenz's Laws of Induction

Faraday's Law of electromagnetic induction:



The induced electromotance ε in any closed circuit is equal to the negative of the time rate of change of the magnetic flux Φ through the circuit.

$$\varepsilon = \frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot\mathbf{dS}$$

In terms of E- and B-fields:

Integral form:

$$\oint_{\partial A} \mathbf{E}\cdot\mathbf{dl} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot\mathbf{dS}$$

Differential form:

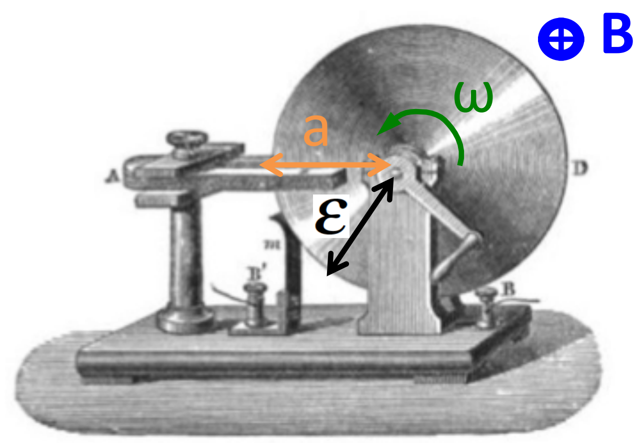
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Lenz's Law:

An induced electromotance always gives rise to a current whose magnetic field opposes the original change in magnetic flux.

3.3. Electromagnetic Induction - Examples

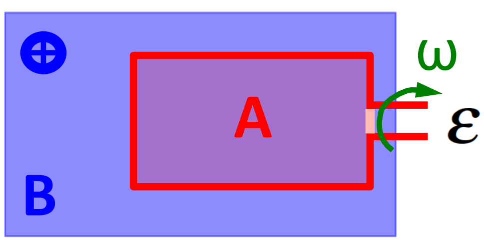
Example 20: the Homopolar Generator (Faraday's disk).



Electric potential (emf) induced between the disk's axis and its rim:

$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{1}{2}\,\omega\, a^2 B$$

Example 21: coil rotation in a B-field.



Electric potential induced in the coil:

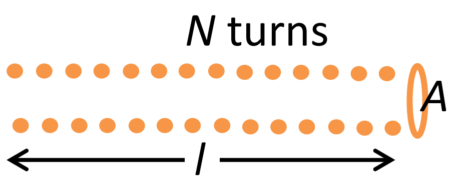
$$\varepsilon = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -N\,A\,B\,\omega\cos(\omega t)$$

3.4. Self-Inductance

Self-inductance L is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.

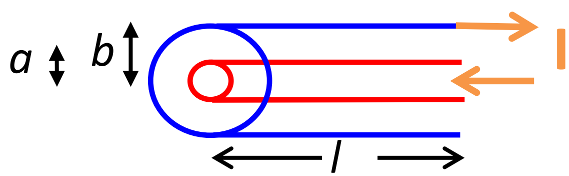
$$L = \frac{\frac{\mathrm{d}\Phi}{\mathrm{d}t}}{\frac{\mathrm{d}I}{\mathrm{d}t}} = \frac{\mathrm{d}\Phi}{\mathrm{d}I} = \frac{-\varepsilon}{\dot{I}}$$

Example 22: Self-inductance of a long coil.



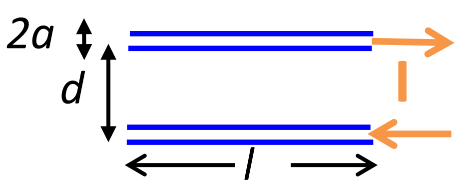
$$L = \frac{\mathrm{d}\Phi}{\mathrm{d}I} = \mu_0 \frac{N^2}{l} A$$

Example 23: Self-inductance of a coaxial cable.



$$L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right) l$$

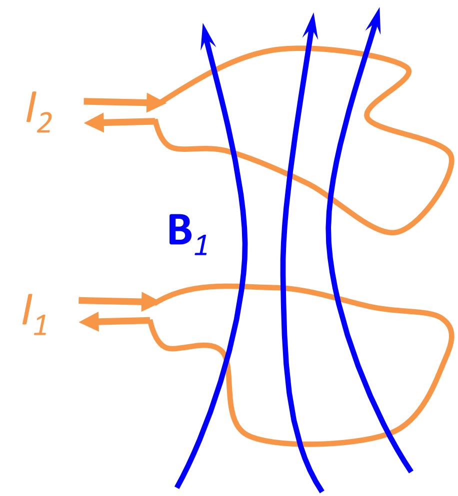
Example 24: Self-inductance of two parallel wires.



$$L = \frac{\mu_0}{\pi} \ln\left(\frac{d-a}{a}\right) l$$

3.4. Mutual Inductance

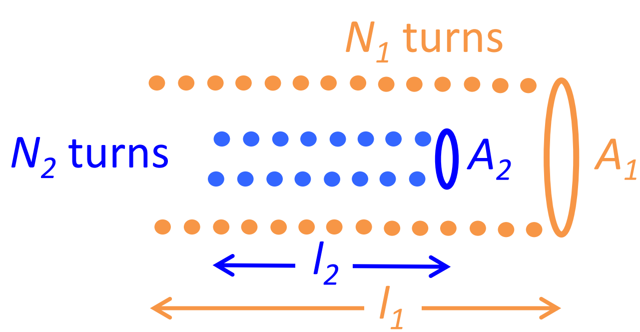
Mutual Inductance M: is the ratio of the voltage (emf) produced in a circuit by self-induction, to the rate of change in current causing the induction.



$$M_{12} = \frac{d\Phi_2}{dI_1} \quad \xleftrightarrow[\text{formula}]{\text{Neumann}} \quad M_{21} = \frac{d\Phi_1}{dI_2}$$

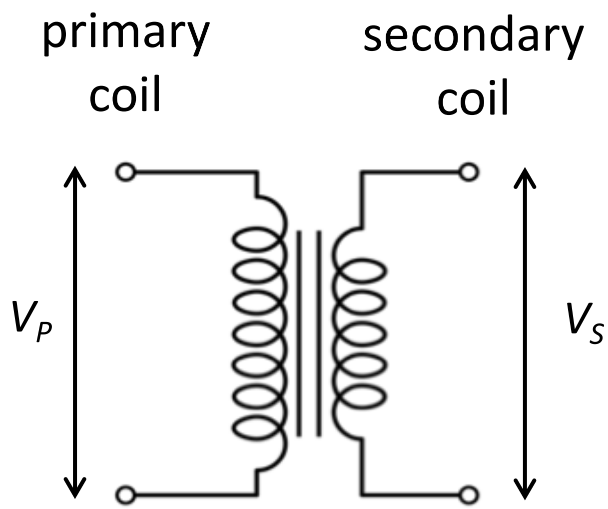
$$M_{12} = M_{21}$$

Example 25: Mutual inductance of two coaxial solenoids.



$$M_{12} = \mu_0 \frac{N_1 N_2}{l_1} A_2$$

3.5. The Transformer



Primary coil creates flux which permeates secondary coil, coupling their voltages:

Voltage Ratio: $$\frac{V_S}{V_P} = \frac{\mathrm{d}\Phi_S}{\mathrm{d}\Phi_P}\frac{N_S}{N_P}$$

Current Ratio: $$\frac{I_S}{I_P} = \frac{\mathrm{d}\Phi_P}{\mathrm{d}\Phi_S}\frac{N_P}{N_S}$$

3.6. Energy of the Magnetic Field

In terms of inductance: $$W_\mathrm{M} = \frac{1}{2}LI^2 = \frac{1}{2}\Phi I$$

$\longleftrightarrow$

In terms of source current: $$W_\mathrm{M} = \frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})\mathrm{d}V$$

$\longleftrightarrow$

In terms of magnetic flux density: $$W_\mathrm{M} = \frac{1}{2\mu_0}\int B^2\mathrm{d}V$$

3.7. Charged Particles in and E- and B-fields

In the presence of both E- and B-fields, a charge experiences the force:

$$\mathbf{F}_{\text{EM}} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right)$$

Example 26: Mass Spectrometer.

A. velocity filter:

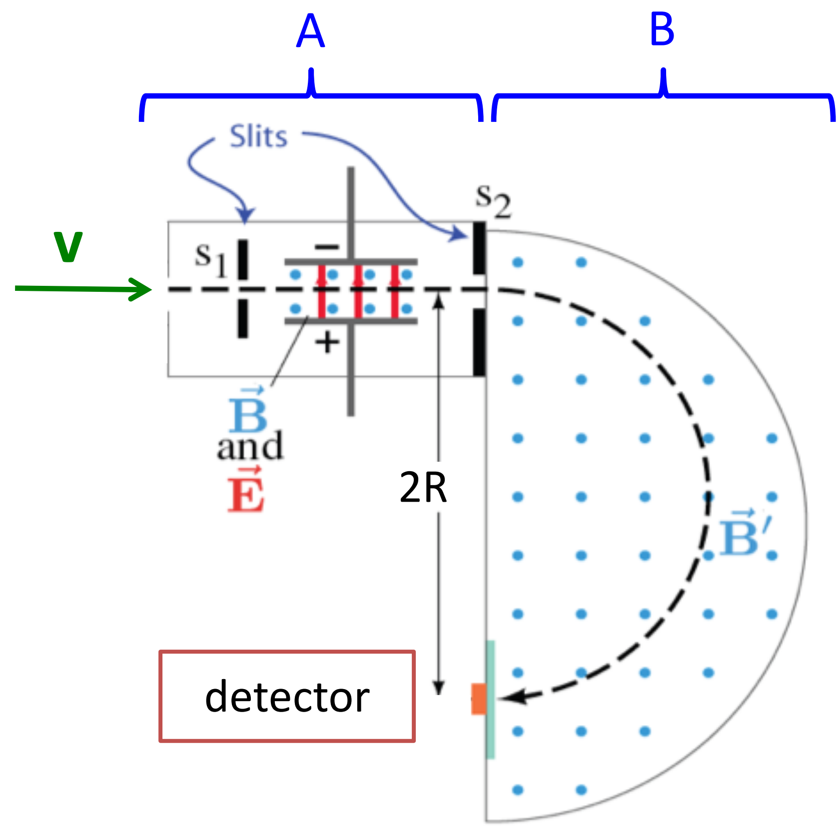
E&B-fields present. Charged particles pass through Stage A if their velocity equals the amplitude ratio:

$$v = \frac{|\mathbf{E}|}{|\mathbf{B}|}$$

B. Filter stage:

Only B-field present. Charged particles are forced on circular path with radius:

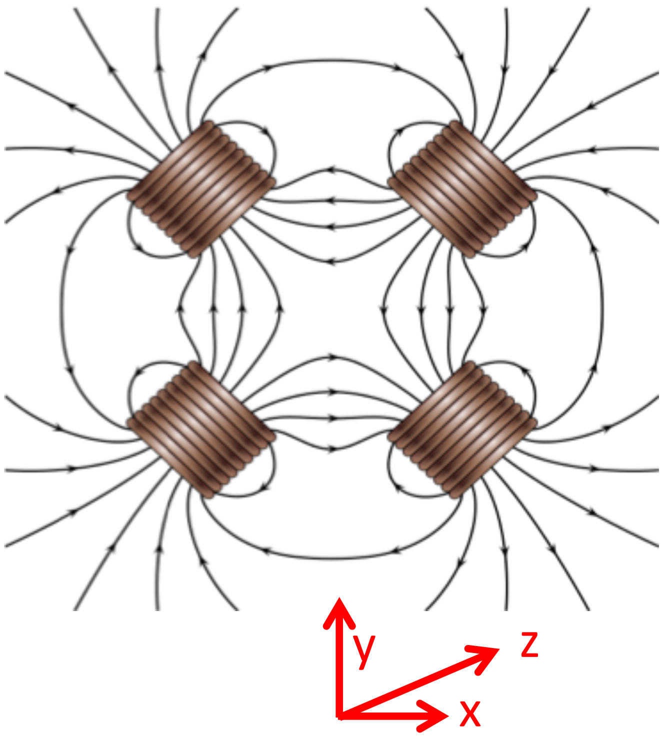
$$R = \frac{mv}{qB}$$



3.7. Charged Particles in and E- and B-fields

Example 27: Magnetic Lens.

$$\mathbf{B} = (Ay, Ax, 0)$$



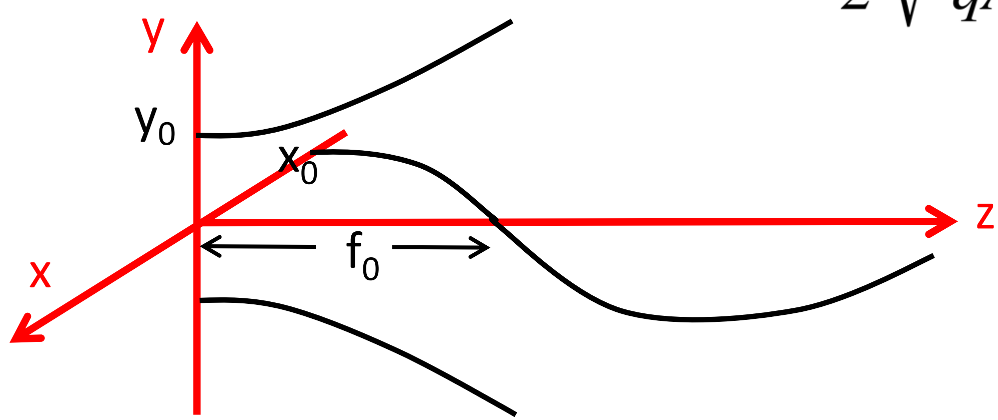
Equation of Motion: $m\ddot{\mathbf{r}} = q\dot{\mathbf{r}} \times \mathbf{B}$

Solutions:

$$y(z) = y_0 \cosh \sqrt{\frac{qA}{vm}} z$$ de-focusing

$$x(z) = x_0 \cos \sqrt{\frac{qA}{vm}} z$$ focusing with

$$f_0 = \frac{\pi}{2}\sqrt{\frac{vm}{qA}}$$



4. Maxwell's Equations and Electromagnetic Waves

4.1. Ampere's Law and the Displacement Current

4.2. Maxwell's Equations

4.3. Electromagnetic Waves in Vacuum

4.4. Energy Flow and The Poynting Vector

Problem Set 5

4.1. Ampere's Law and the Displacement Current

Ampere's law does not comply with the Equation of Continuity:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

apply div:

$$\underbrace{\nabla \cdot (\nabla \times \mathbf{B})}_{\substack{=0 \\ \text{always}}} = \mu_0 \underbrace{\nabla \cdot \mathbf{J}}_{= -\frac{\partial \rho}{\partial t}}$$

$$= 0 \quad \text{only for statics!}$$

This lack of charge conservation is unphysical! As a solution, add a so-called "displacement current" to **J**, which will ensure compliance with the equation of continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\varepsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot (\underbrace{\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}_{\text{displacement current } \mathbf{J}_D})$$

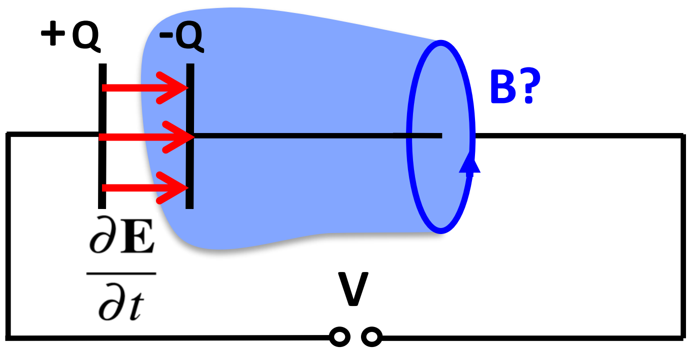
Obtain **Ampere's law with "displacement current"**:

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t})$$

4.1. Ampere's Law and the Displacement Current

Ampere's law with displacement current: $$\oint_{\partial A} \mathbf{B} \cdot d\mathbf{l} = \mu_0 (I + I_D)$$

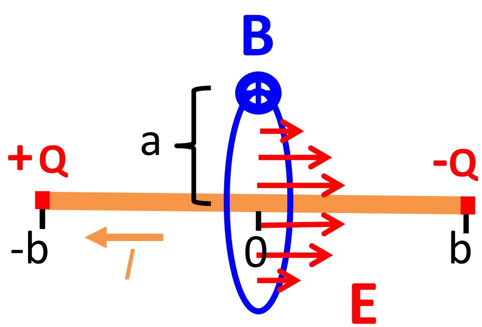
Example 28: A charging capacitor and Ampere's law.



Calculate **B** along Amperean loop. But area bounded by the loop could e.g. be the plane surface enclosed, or a "bulged" surface passing through the capacitor.

Add displacement current $I_D = \varepsilon_0 A \frac{\partial \mathbf{E}}{\partial t}$ account for changing E-field between plates.

Example 29: Magnetic field of a short, current-carrying wire – revisited.



Because the wire is short, charge builds up at the end, causing a time-varying electric field through the area bounded by the Amperean loop.

Add displacement current:

$$I_D = \frac{\partial q}{\partial t}\left[\frac{b}{\sqrt{b^2 + a^2}} - 1\right]$$

4.2. Maxwell's Equations

$$\oint_V \mathbf{E}\cdot\mathbf{dA} = \frac{Q_V}{\varepsilon_0} \leftrightarrow \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$$

Gauss' law: Charge generates an electric field. Electric field lines begin and end on charge.

$$\oint_{\partial A} \mathbf{E}\cdot\mathbf{dl} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{B}\cdot\mathbf{dS}$$

$$\leftrightarrow \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

Faraday's law: time-varying magnetic fields create electric fields (induction).

$$\oint_V \mathbf{B}\cdot\mathbf{dA} = 0 \leftrightarrow \nabla\cdot\mathbf{B} = 0$$

There are **no magnetic monopoles.** Magnetic field lines form closed loops.

$$\oint_{\partial A} \mathbf{B}\cdot\mathbf{dl} = \mu_0 I + \mu_0\varepsilon_0\frac{\mathrm{d}}{\mathrm{d}t}\int_A \mathbf{E}\cdot\mathbf{dS}$$

$$\leftrightarrow \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

Ampere's law including displacement current: electric currents and time-varying electric fields generate magnetic fields.

4.3. Electromagnetic Waves in Vacuum

In vacuum, free of charge or currents (ρ, **J** = 0):

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla^2 \mathbf{E} = \varepsilon_0 \mu_0 \ddot{\mathbf{E}}$$

$$\nabla^2 \mathbf{B} = \varepsilon_0 \mu_0 \ddot{\mathbf{B}}$$

Wave equations in **E, B**!

Electromagnetic waves propagate in free space:

Plane EM wave fronts: $\mathbf{E} = \mathbf{E_0} \exp\{i(\omega t - \mathbf{k} \cdot \mathbf{r})\}$ with wavelength $\lambda = \frac{2\pi}{k}$

Propagation velocity of wave fronts: $c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 3 \times 10^8 \, \mathrm{m\,s^{-1}}$

Relationship between E and B:
(in phase and mutually orthogonal
with wave vector **k**)

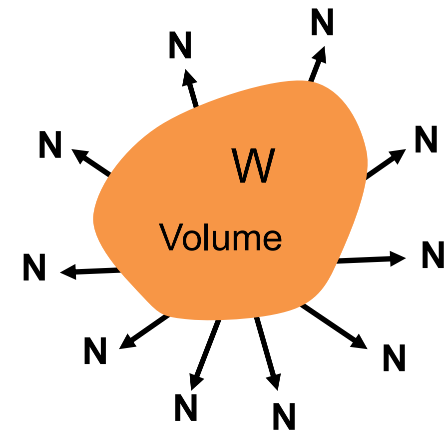
$$\mathbf{B} = \frac{\mathbf{k}}{\omega} \times \mathbf{E} \qquad \mathbf{E} = -c^2 \frac{\mathbf{k}}{\omega} \times \mathbf{B} \qquad \frac{|\mathbf{E}|}{|\mathbf{B}|} = c$$

Impedance of free space: $Z = \frac{|\mathbf{E}|}{|\mathbf{B}|/\mu_0} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\,\Omega$

4.4. Energy Flow and the Poynting Vector

Total electromagnetic energy *W* contained in volume *V*:

$$W = \int_V \frac{1}{2} \underbrace{\left(\varepsilon_0 \mathbf{E} \cdot \mathbf{E} + \frac{1}{\mu_0} \mathbf{B} \cdot \mathbf{B} \right)}_{\text{energy density } u = \frac{\mathrm{d}W}{\mathrm{d}V}} \mathrm{d}V$$



$$\boxed{\frac{\mathrm{d}W}{\mathrm{d}t} = - \oint_{\partial V} \mathbf{N} \cdot \mathrm{d}\mathbf{S} \quad \text{with} \quad \mathbf{N} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{Poynting vector}}$$

Energy flow rate out of volume V

Power per unit area through area bounding V

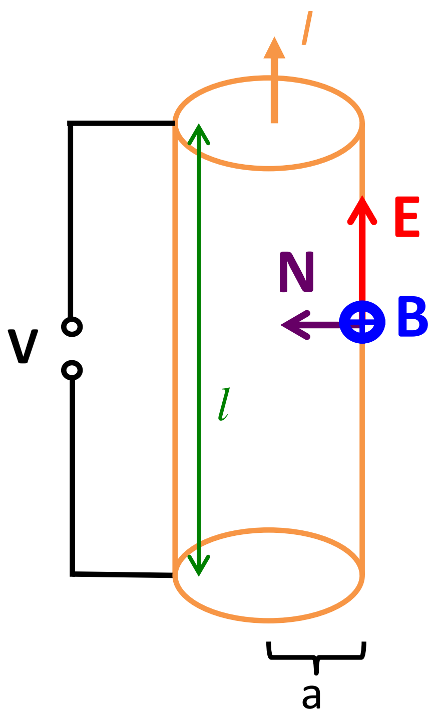
$[N]=W/m^2$

The intensity *I* of an EM wave is given by the time-average over the magnitude of the Poynting vector:

$$I = \langle |\mathbf{N}| \rangle = \frac{1}{2} \varepsilon_0 \, c \, \mathbf{E_0}^2$$

4.4. Energy Flow and the Poynting Vector

Example 30: Poynting vector for a long resistive rod.



Calculate Poynting vector at the surface of a wire with applied potential difference V and current I:

$$\mathbf{N} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$$

$$N = \frac{1}{\mu_0}\frac{V}{l}\frac{\mu_0 I}{2\pi a} = \frac{VI}{2\pi l a}$$

power dissipated in wire

surface of wire

Total power dissipated in wire is equal to *V I*.