First-Year Electromagnetism: Problem Set 3

Hilary Term 2018, Prof LM Herz

D. Capacitance and Electric-Field Energy

D.0 Background. State the definition of capacitance, and derive the electric field energy stored inside a capacitor.

D.1 Spherical and cylindrical capacitors. Use Gauss' law to derive expressions for the capacitance C of:

(a) two concentric spherical conducting shells with radii R_1 and R_2.

(b) two very long, coaxial cylindrical conducting shells with radii R_1 and R_2 and axial length L.

Show that as $|R_2 - R_1|$ becomes small, the systems described in (a) and (b) become equivalent to a parallel-plate capacitor.

D.2 Charge distribution inside a parallel-plate capacitor. A capacitor consists of two parallel large conducting planes separated by a distance d. The space between the plates is filled with a uniform, immobile space charge of density ρ. Find the magnitude of the electric field at a distance x from the positive plate when a potential difference V is applied to the capacitor. Discuss whether the capacitance of the capacitor is affected by the presence of the space charge.

[Answer: $-(V/d) + \rho(d - 2x)/2\epsilon_0$]

D.3 Air breakdown thresholds inside a cylindrical capacitor. A capacitor consists of two air-spaced concentric cylinders, similar to that described in Problem D.1(b). The outer radius is fixed at b=10 mm, while the inner radius a is variable. Electric-field induced breakdown of air will occur for field strengths exceeding E_b=3 MVm^{-1}. Show that

(a) $a = b/e$ is required for maximixed potential difference across the capacitor

(b) $a = b/\sqrt{e}$ is needed for maximized energy per unit length stored in the capacitor

in the absence of air breakdown. [Here e is Euler's number.]

D.4 Forces between capacitor plates. Two parallel plates of a capacitor are spaced 10 mm apart, have an effective area of 0.01 m^2 each, and a potential difference of 600 V maintained between them through a connected battery.

(a) Determine the force between the two plates by considering the energy stored or supplied to the system. [Answer: 1.59×10^{-4} N]

(b) Find the work done when the plates are pulled apart slowly to a separation of 20 mm

  i. while the potential difference is maintained at 600 V, and

  ii. while the plates are charged to 600 V and isolated before separation.

(c) For case (i), show that the energy stored in the field between the plates is halved when they are slowly pulled apart. Yet work is done in pulling them apart against the attractive force between them. Explain.

E. Magnetostatics

E.0 Background. Describe how the laws of Biot-Savart and Ampère may be used to calculate the magnetic field generated by an electrical current. Explain how you would decide which of the two was the most appropriate to use for a given situation.

E.1 Magnetic fields from straight current segments and polygons.

(a) A straight wire of length $2b$ carries a current I. Find the magnitude of the magnetic field H at a distance a from the wire along its perpendicular bisector.

(b) N equal straight wires, carrying current I, form a closed regular polygon circumscribed about a circle of radius a. Calculate the magnetic field at the centre of the polygon and show that it takes the value at the centre of a circular loop of radius a when $N \to \infty$. [Answer: $NI\sin(\pi/N)/2\pi a$]

E.2 On-axis magnetic field of a coil and of a pair of Helmholtz coils

(a) A circular coil of N turns has radius a and negligible cross-section. Show that when a current I passes through it, the magnitude of the magnetic field H at a point on its axis a distance x from the centre of the coil is given by:

$$H = \frac{NIa^2}{2(a^2 + x^2)^{3/2}}$$

(b) Hence show that when the current passes in the same direction in two identical coaxial coils separated by a distance equal to their radius, there is a small region on the axis mid-way between the coils in which the first, second and third differential coefficients of the field with respect to x are all zero. What is the practical importance of such a Helmholtz coil system?

E.3 In-plane magnetic field of a coil. A plane circular coil of N turns (of negligible cross-section) carries current I and has radius a. Calculate the magnetic flux density at a point lying in the plane of the coil, a distance $r \gg a$ from the centre of the coil. Show that the same result is obtained by substituting a magnetic dipole for the coil. [Answer: $\mu_0 NIa^2/4r^3$]

E.4 Magnetic field inside a solenoid A solenoid of radius a and length l is uniformly wound with n closely spaced turns of wire per unit axial length. A current I flows through the wire. The magnetic flux density on the axis of the solenoid is measured to be $\sqrt{2}$ times as large at the centre as it is at the ends of the solenoid. What is the ratio of l to a?

E.5 Magnetic field of a long cylindrical wire. A uniform current density J exists along the z-direction between cylinders $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$.

(a) Determine the magnitude of the magnetic field in the regions (i) $r<a$, (ii) $a<r<b$, and (iii) $r>b$.

(b) Sketch the dependence of the magnetic field magnitude on r.