Revision Lecture on

ELECTROMAGNETISM (CP2)

- Electrostatics
- Magnetostatics
- Induction
- EM Waves

… based on previous years' Prelims questions

State Coulomb’s Law. Show how **E** field may be defined. What is meant by **E** is a conservative field?

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}\hat{\mathbf{r}}_{12}$$

Electric field: $\mathbf{E} = \lim_{q\to 0}\frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

Conservative field: $\nabla\times\mathbf{E} = 0$ and $\int \mathbf{E}\cdot d\mathbf{l}$ is path-independent. Therefore, can define a potential.

A thundercloud with charges +40As at 10 km height and −40As at 6 km. Find the E-field on the ground.

Use method of image charges. Mirror the above to below the surface, with +40 As at depth 6 km and −40 As at depth 10 km.

$$E = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\left(10^4\,\mathrm{m}\right)^2} - \frac{1}{\left(6\times10^3\,\mathrm{m}\right)^2} - \frac{1}{\left(6\times10^3\,\mathrm{m}\right)^2} + \frac{1}{\left(10^4\,\mathrm{m}\right)^2}\right]$$

$$= \frac{2\cdot 40\mathrm{As} \quad \mathrm{Vm}}{4\pi\cdot 8.854\times10^{-12}\,\mathrm{As}}\left[\frac{1}{3.6\times10^7\,\mathrm{m}^2} - \frac{1}{10^8\,\mathrm{m}^2}\right] = 12{,}780\,\frac{\mathrm{V}}{\mathrm{m}}$$

Field point upwards.

An array of localised charges q_i experience potentials V_i as a result of their mutual interaction. Show that their mutual electrostatic energy, *W*, is given by $W = \frac{1}{2}\sum_i q_i V_i$.

Potential energy of charge *q* in potential *V*:

$$W = \int \mathbf{F}\cdot d\mathbf{l} = -q\cdot\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l} = q\cdot V(r)$$

Potential V_i due to all other charges:

$$V_i = \sum_j \frac{q_j}{4\pi\varepsilon_0}\cdot\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

For total PE, sum over all charges. However, each charge appears twice:

$$\frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{4\pi\varepsilon_0}\cdot\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

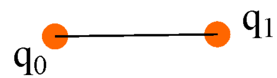
$$W = \tfrac{1}{2}\sum_i q_i V_i$$

Alternative: Assemble Charge Configuration

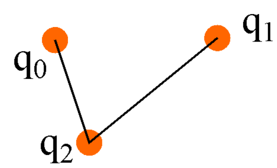
$$W = \frac{1}{4\pi\varepsilon_0}\Big[\, 0 + \frac{q_0 q_1}{r_{01}} + \frac{q_0 q_2}{r_{02}} + \frac{q_1 q_2}{r_{12}} + \frac{q_0 q_3}{r_{03}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \Big]$$

No penalty for charge q_0

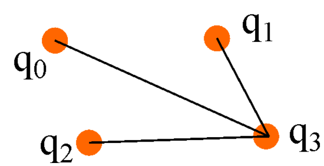
q_1 in potential due to q_0



q_2 in potential of q_0 and q_1



q_3 in pot. of q_0, q_1 and q_2



Half the links compared with:

0 1 2 3

Thus:

$$W = \tfrac{1}{2}\sum_i q_i V_i$$

A sphere of radius *a* is located at a large distance from its surroundings which define the zero of potential. It carries a total charge *q*. Determine the potential on its surface and the electrostatic energy.

$$V(r) = -\int_{\infty}^{r} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = \frac{q}{4\pi\varepsilon_0 r} \qquad W = \frac{1}{2}\sum_i q_i V_i \quad \text{or} \quad \int V \mathrm{d}q$$

Shell:

$$V = \frac{q}{4\pi\varepsilon_0 a} \quad \text{and} \quad W = \frac{1}{2} q \frac{q}{4\pi\varepsilon_0 a} = \frac{q^2}{8\pi\varepsilon_0 a}$$

(alternative: $W = \int V \mathrm{d}q = \int \frac{q\mathrm{d}q}{4\pi\varepsilon_0 a} = \frac{q^2}{8\pi\varepsilon_0 a}$)

Uniformly charged sphere:

Potential of the surface is the same for sphere and shell (Gauss; same charge inside)

$$W = \int V \mathrm{d}q = \int V \rho \, \mathrm{d}^3 r$$

$$W = \int \frac{r^3}{a^3} \; \frac{q}{4\pi\varepsilon_0 r} \; \frac{q}{\frac{4\pi}{3} a^3} \; r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \, \mathrm{d}r$$

fraction inside

potential

charge density

volume element

$$W = 3 \frac{q^2}{4\pi\varepsilon_0} \int \frac{r^4}{a^6} \mathrm{d}r = \frac{3}{5} \frac{q^2}{4\pi\varepsilon_0 a}$$

With Gauss' Law:
$$\oiint \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\cdot\iiint \rho dV$$

$$\oiint E_r dS = E_r \cdot 4\pi r^2 = \tfrac{1}{\varepsilon_0}\cdot\begin{cases}\tfrac{4\pi}{3}R^3\rho_0 & \text{for} \quad r \geq R \\ \tfrac{4\pi}{3}r^3\rho_0 & \text{for} \quad r < R\end{cases}$$

$$E_r = \frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r^2} \quad \text{for } r \geq R \qquad \text{and} \qquad E_r = \frac{\rho_0}{3\varepsilon_0}\cdot r \quad \text{for } r < R$$

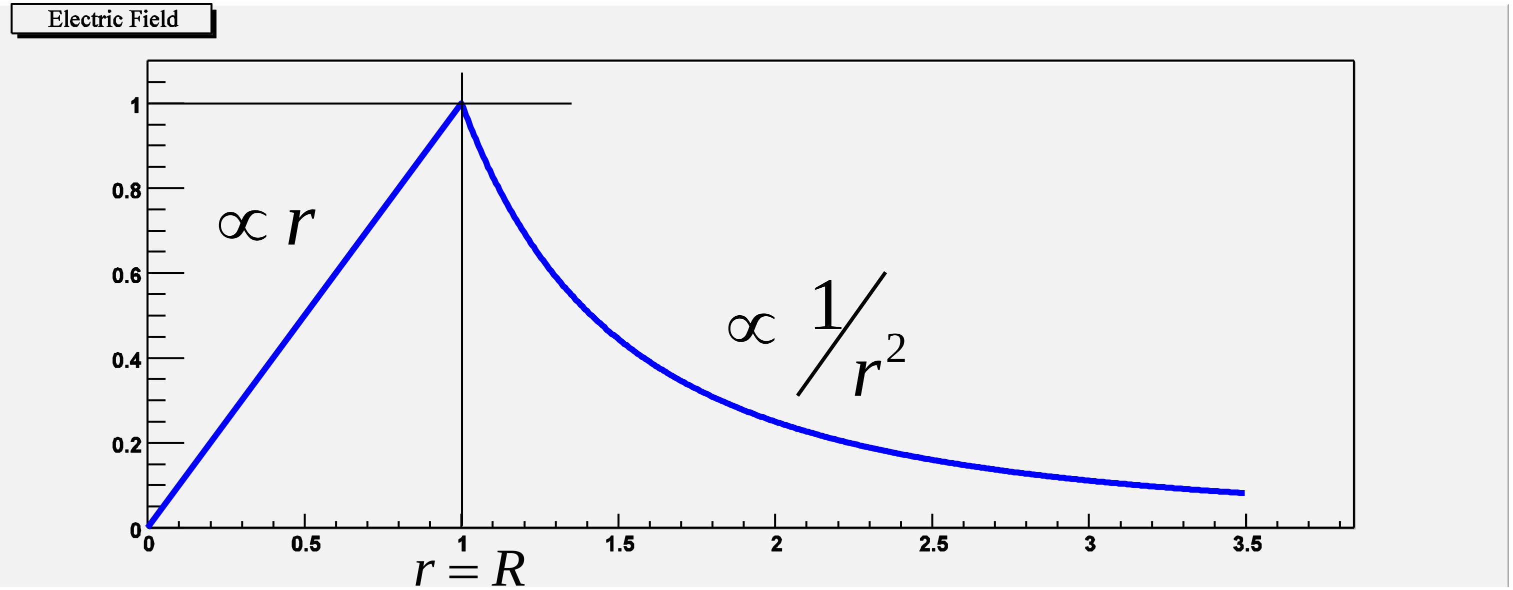
$$V_{out} = -\int_\infty^r E_r dr' = -\frac{\rho_0}{3\varepsilon_0}R^3\cdot\left[-\frac{1}{r'}\right]_\infty^r = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\cdot\frac{R^3}{r}}}$$

on sphere $(r = R)$: $\quad V_S = \dfrac{\rho_0}{3\varepsilon_0}R^2$

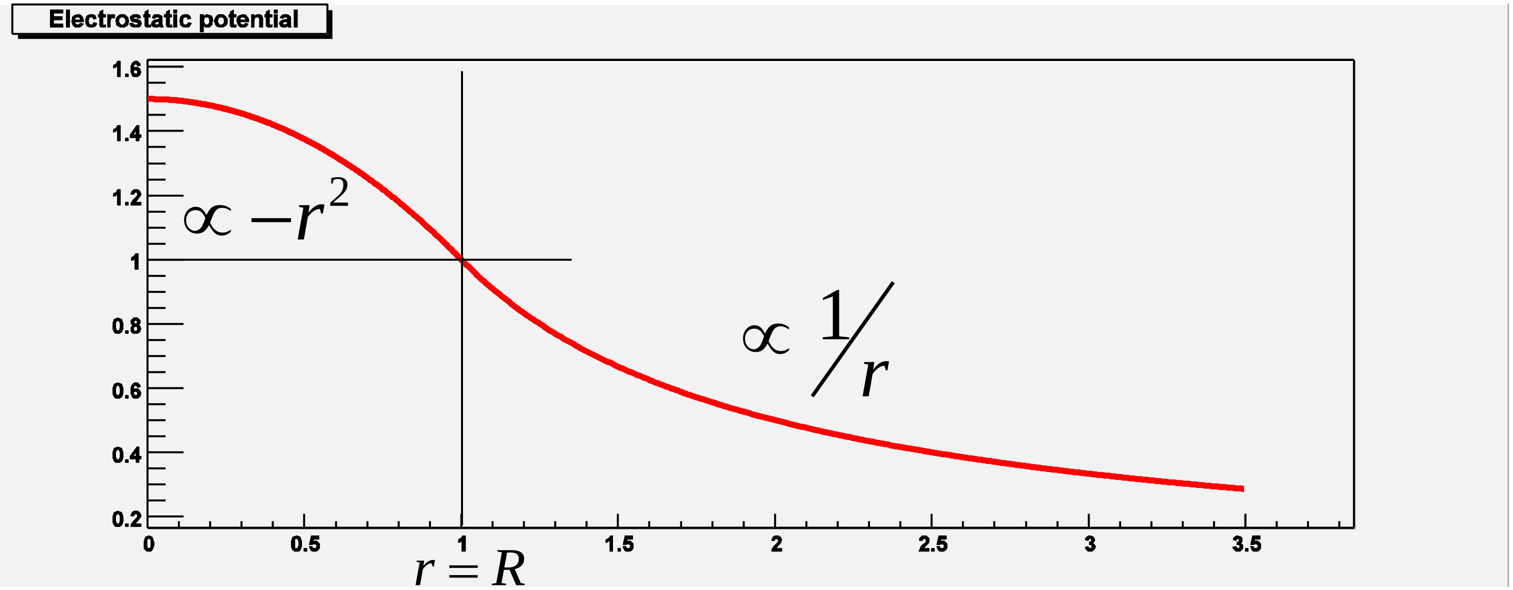
$$V_{ins} = V_S - \int_R^r E_r dr' = \frac{\rho_0}{3\varepsilon_0}\left[R^2 - \tfrac{1}{2}r^2 + \tfrac{1}{2}R^2\right] = \underline{\underline{\frac{\rho_0}{3\varepsilon_0}\left[\tfrac{3}{2}R^2 - \tfrac{1}{2}r^2\right]}}$$

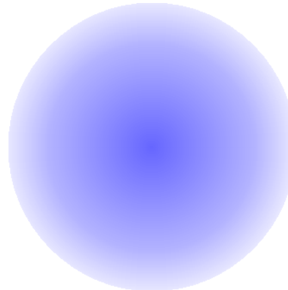
E-field and potential V as function of r

$$E_{\max} = \frac{\rho_0}{3\varepsilon_0} R$$



$$V_S = \frac{\rho_0}{3\varepsilon_0} R^2$$



Electron cloud: 

$$\rho(r) = -\frac{e}{\pi a_0^3} \cdot \exp\left(-\frac{2r}{a_0}\right)$$

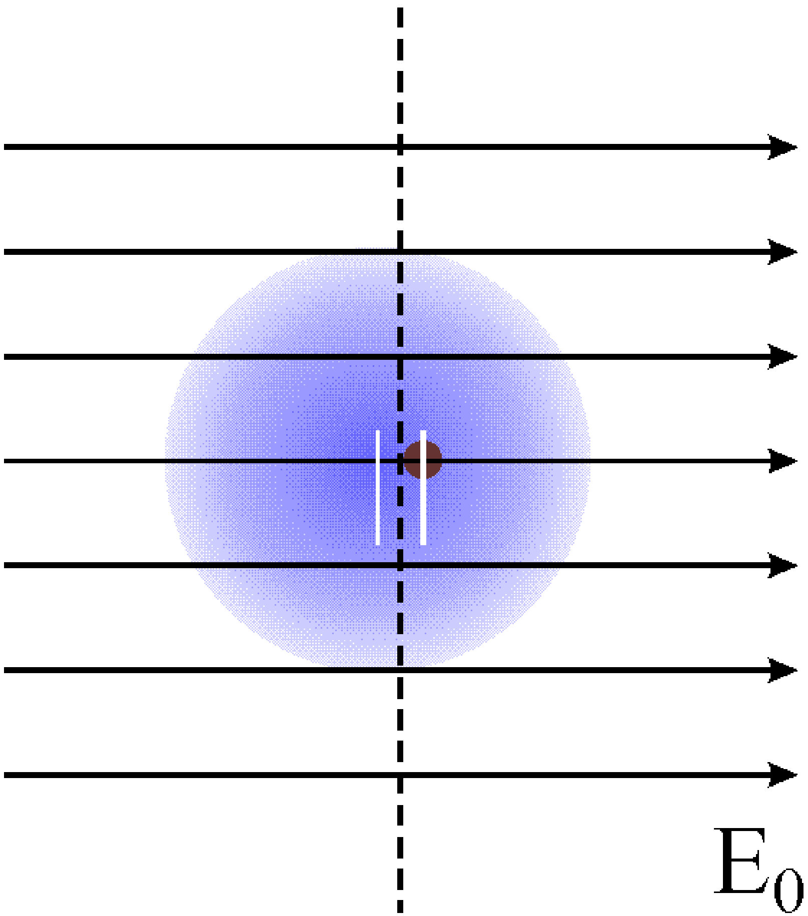
$$E_r = \frac{1}{4\pi\varepsilon_0 r^2} \cdot \left[-\frac{e}{\pi a_0^3} \cdot \iiint \exp\left(-\frac{2r'}{a_0}\right) r'^2 \sin\theta d\theta d\varphi dr' \right]$$

$$\int_0^r x^2 \exp(ax) dx = \tfrac{1}{a} x^2 e^{ax} \Big|_0^r - \tfrac{2}{a^2} x e^{ax} \Big|_0^r + \tfrac{2}{a^3} e^{ax} \Big|_0^r$$

here: $a = -\dfrac{2}{a_0}$ and $\iint \sin\theta d\theta d\varphi = 4\pi$

$$E_r = \frac{e}{4\pi\varepsilon_0} \left\{ \frac{\exp(-2r/a_0) - 1}{r^2} + \frac{2\exp(-2r/a_0)}{a_0 r} + \frac{2\exp(-2r/a_0)}{a_0^2} \right\}$$

Explain qualitatively what happens when such an atom is placed in a steady, uniform electric field, of strength E_0.

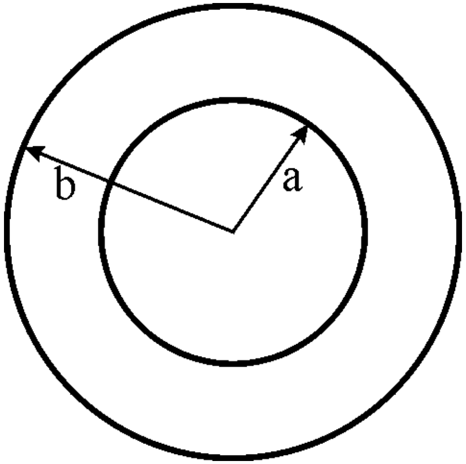


Centres of gravity of the positive nucleus and the negative electron charge distribution shift.

The atom exhibits an electric dipole moment.

Gauss's Theorem in vacuo: $\oiint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \cdot \iiint \rho \, dV$

Calculate the capacitance for a spherical capacitor: $C = \frac{Q}{V}$



$$E_r \cdot 4\pi r^2 = {}^1\!/_{\varepsilon_0} \cdot Q$$

$$V = -\int_b^a E_r dr = \frac{Q}{4\pi\varepsilon_0} \cdot \left[\frac{1}{r}\right]_b^a = \frac{Q \cdot (b-a)}{4\pi\varepsilon_0 ab}$$

$$\underline{\underline{C = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a}}}$$

The inner sphere is raised to a potential V and then isolated, the outer sphere being earthed. The outer sphere is then removed. Find the resulting potential of the remaining sphere.

Charge stored on inner sphere: $$Q = 4\pi\varepsilon_0 \cdot \frac{ab}{b-a} \cdot V$$

Field of remaining sphere: $$E_r = \frac{Q}{4\pi\varepsilon_0 r^2}$$

$$\underline{\underline{V'}} = -\int_{\infty}^{a} E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_{\infty}^{a} \frac{1}{r^2} dr = \frac{Q}{4\pi\varepsilon_0 a} = \underline{\underline{\frac{b}{b-a} \cdot V}}$$

Maximum potential to which the inner sphere can be charged to:

$$E_{\max} = 3000\,\mathrm{V/mm} \qquad a = 0.9\,\mathrm{m} \qquad b = 1.0\,\mathrm{m}$$

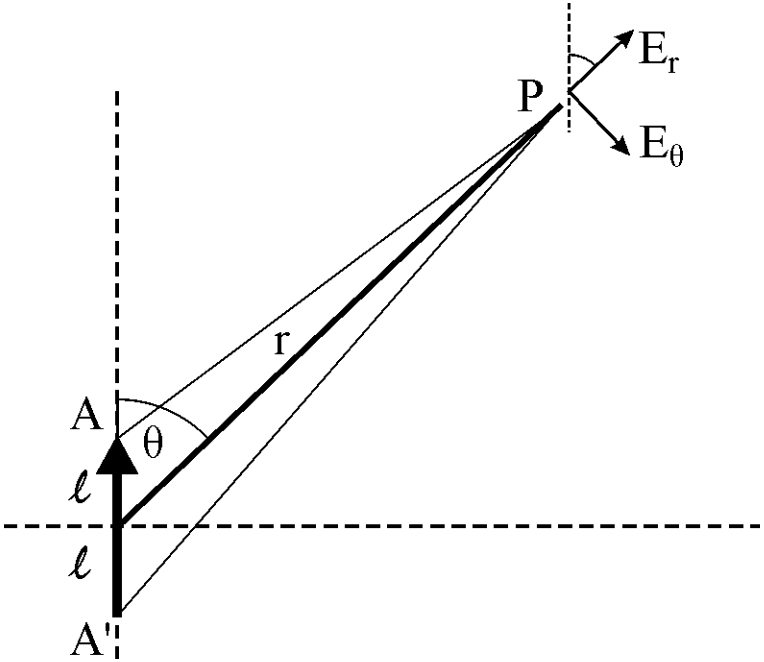
E is maximal when r is minimal: Consider E(a)

$$E_r(a) = \frac{ab}{b-a} \cdot V \cdot \frac{1}{a^2} = V \cdot \frac{b}{a} \cdot \frac{1}{b-a}$$

$$\underline{\underline{V_{\max}}} = E_{\max} \cdot \frac{a(b-a)}{b} = 3 \cdot 10^6 \,\frac{\mathrm{V}}{\mathrm{m}} \cdot \frac{0.9\,\mathrm{m} \cdot 0.1\,\mathrm{m}}{1\,\mathrm{m}} = \underline{\underline{2.7 \cdot 10^5\,\mathrm{V}}}$$

The electrostatic potential of a dipole:

Charges +q at A and –q at A'



$$\overline{AP}^2 = r^2 + \ell^2 - 2r\ell\cos\theta$$

$$\overline{AP'}^2 = r^2 + \ell^2 + 2r\ell\cos\theta$$

$$V_P = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{AP}} - \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{\overline{AP'}}$$

$$\frac{1}{\overline{AP}} = \frac{1}{r}\cdot\left[1+\left(\ell/r\right)^2 - 2\,\ell/r\cos\theta\right]^{-1/2} \approx \frac{1}{r}\cdot\left[1+\ell/r\cos\theta+\dots\right]$$

$$\frac{1}{\overline{AP'}} \approx \frac{1}{r}\cdot\left[1-\ell/r\cos\theta+\dots\right]$$

$$\text{so:}\ \ V_P = \frac{q}{4\pi\varepsilon_0 r}\cdot\left[1+\ell\!/_{r}\cos\theta - 1 + \ell\!/_{r}\cos\theta\right] = \frac{2q\ell}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}\cos\theta$$

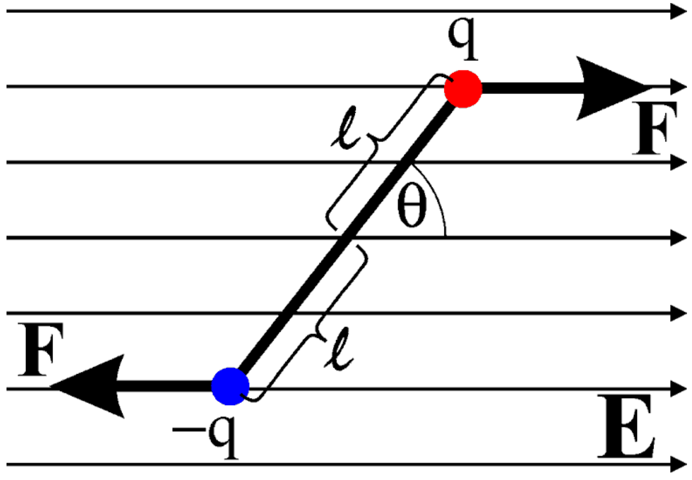
$$\underline{\underline{V_P = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}}$$

The radial and tangential components of the E-field:

$$\mathbf{E} = -grad\left(V_P\right); \qquad E_r = -\frac{\partial V_P}{\partial r} \quad \text{and} \quad E_\theta = -\frac{1}{r}\cdot\frac{\partial V_P}{\partial \theta}$$

$$\underline{\underline{E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}}} \qquad \text{and} \qquad \underline{\underline{E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}}}$$

Show that the torque exerted on a dipole by a uniform electric field **E** is **p**×**E**.



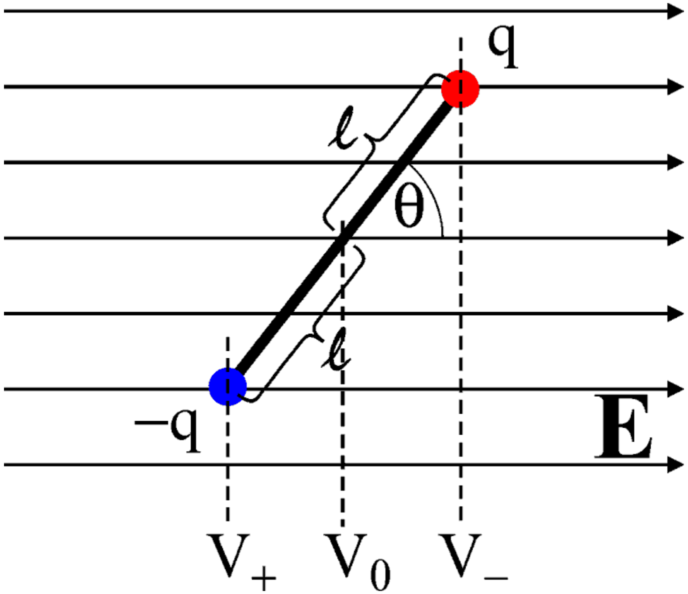
$$T = r \cdot F = 2\ell \sin\theta \cdot F =$$

$$= 2q\ell E \sin\theta$$

with $p = 2q\ell$:

$$\mathbf{T} = \mathbf{p} \times \mathbf{E}$$

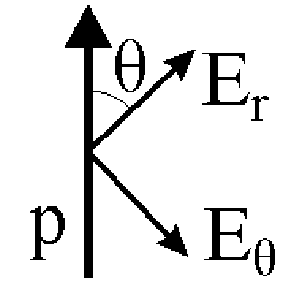
Calculate the work done in bringing a dipole of equal magnitude from infinity to a distance r from the first along the normal to its axis.



$$U_E = (-q)\cdot V_+ + q\cdot V_-$$
$$= -q\cdot(V_+ - V_-)$$
$$V = V_0 + \nabla V\cdot\mathbf{r} = V_0 - \mathbf{E}\cdot\mathbf{r}$$
$$V_+ - V_- = -\mathbf{E}\cdot(\mathbf{r}_{q-} - \mathbf{r}_{q+})$$
$$U_E = q\mathbf{E}\cdot(-2\mathbf{l}) = -\mathbf{p}\cdot\mathbf{E}$$

$$\underline{\underline{U_E}} = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta = 2q\ell\cdot\frac{2q\ell}{4\pi\varepsilon_0 r^3}\cdot\cos\theta = \underline{\underline{\frac{4q^2\ell^2}{4\pi\varepsilon_0 r^3}\cos\theta}}$$

Find the angle θ for which **E**(r, θ) is in a direction normal to the axis of the dipole.



Find angle for which $\mathbf{p} \cdot \mathbf{E} = p_z \cdot E_z = 0$

$$E_z = E_r \cdot \cos\theta - E_\theta \cdot \sin\theta = 0 \quad \text{thus} \quad \frac{2p\cos^2\theta}{4\pi\varepsilon_0 r^3} - \frac{p\sin^2\theta}{4\pi\varepsilon_0 r^3} = 0$$

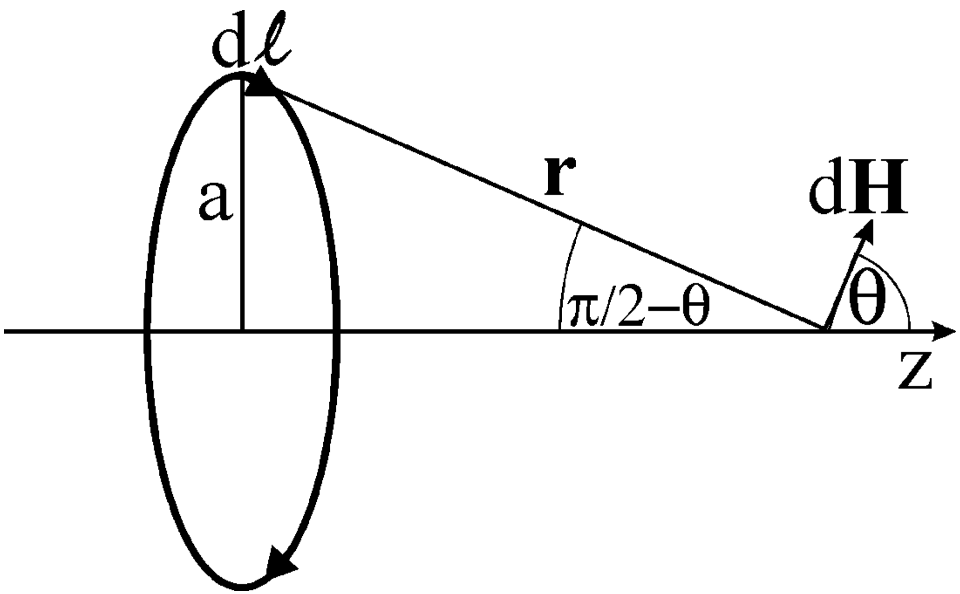
$$2\cos^2\theta = \sin^2\theta \qquad \text{and} \qquad \tan\theta = \pm\sqrt{2} \quad \text{or} \quad \theta = \pm 54.73^o$$

Second dipole placed at θ=0 and θ=π/2:

$\theta = 0$	$E_r = \dfrac{2p}{4\pi\varepsilon_0 r^3}$	$E_\theta = 0$	p_2 parallel
$\theta = \dfrac{\pi}{2}$	$E_r = 0$	$E_\theta = \dfrac{p}{4\pi\varepsilon_0 r^3}$	p_2 anti-parallel

State the law of Biot-Savart: $$\mathbf{dB} = \mu_0 I \cdot \frac{\mathbf{dl} \times \mathbf{r}}{4\pi r^3}$$

Find the magnitude of **B** on axis for a coil of *n* turns



Symmetry:

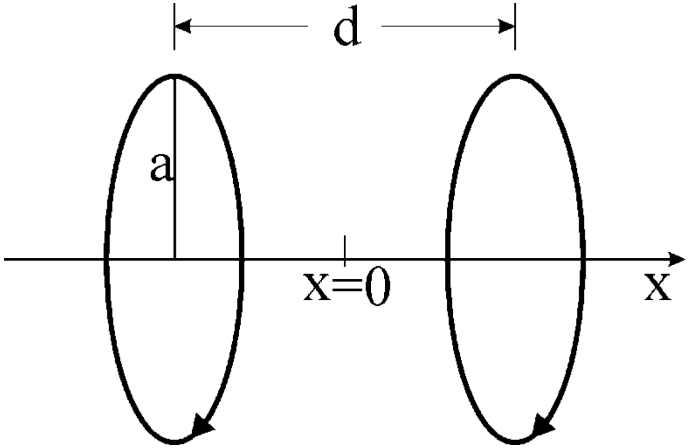
d**B** has z-component only

⊥-components cancel

And also: d**l** ⊥ **r**

$$\underline{\underline{B}} = B_z = \int \frac{\mu_0 n I}{4\pi r^2} \cdot \underbrace{\left| \frac{\mathbf{dl} \times \mathbf{r}}{r} \right|}_{a d\varphi} \cdot \underset{a/r}{\cos\theta} = \int_0^{2\pi} \frac{\mu_0 n I a^2 d\varphi}{4\pi r^3} = \underline{\underline{\frac{\mu_0 n I a^2}{2\left(z^2 + a^2\right)^{3/2}}}}$$

Two such coils are placed a distance *d* apart on the same axis. Find B as function of x.



$$B'(x) = \frac{\mu_0 n I a^2}{2} \cdot \left[\frac{1}{\left(a^2 + \left(\frac{d}{2} + x\right)^2\right)^{3/2}} + \frac{1}{\left(a^2 + \left(\frac{d}{2} - x\right)^2\right)^{3/2}} \right]$$

Show that the derivative of B' is 0 for x=0

$$\left(a^2 + \left(\tfrac{d}{2} \pm x\right)^2\right)^{-3/2} \xrightarrow{\frac{d}{dx}} -\frac{3}{2}\left(a^2 + \left(\tfrac{d}{2} \pm x\right)^2\right)^{-5/2} \cdot 2\left(\tfrac{d}{2} \pm x\right) \cdot (\pm 1)$$

which is $\pm$ the same, when $x = 0$, hence: $\underline{\underline{\frac{dB'}{dx}(0) = 0}}$

Find the value of d for which the second derivative of B'(0) is 0.

$$\partial_x B' \propto -3\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-5/2}\left(\tfrac{d}{2}+x\right)+3\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-5/2}\left(\tfrac{d}{2}-x\right)$$

$$\partial_x^2 B' \propto \quad -3\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-5/2} \quad + \quad 15\left(a^2+\left(\tfrac{d}{2}+x\right)^2\right)^{-7/2}\left(\tfrac{d}{2}+x\right)^2$$

$$-3\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-5/2} \quad + \quad 15\left(a^2+\left(\tfrac{d}{2}-x\right)^2\right)^{-7/2}\left(\tfrac{d}{2}-x\right)^2$$

$$\partial_x^2 B'(0) \propto -3\cdot\frac{2}{\left(a^2+\left(\tfrac{d}{2}\right)^2\right)^{7/2}}\cdot\left[\left(a^2+\left(\tfrac{d}{2}\right)^2\right)-5\left(\tfrac{d}{2}\right)^2\right]$$

$$a^2-4\left(\tfrac{d}{2}\right)^2=0 \qquad\qquad \underline{\underline{d=a}}$$

Show that the variation of B between the coils is <6%

$$B(x)=\frac{\mu_0 nI}{2a}\cdot\left[\frac{1}{\left(1+\left(\frac{1}{2}+\frac{x}{d}\right)^2\right)^{3/2}}+\frac{1}{\left(1+\left(\frac{1}{2}-\frac{x}{d}\right)^2\right)^{3/2}}\right]$$

$$B(0)=\frac{\mu_0 nI}{2a}\cdot\frac{2}{\left(5/4\right)^{3/2}}$$

$$B\left(\frac{d}{2}\right)=\frac{\mu_0 nI}{2a}\cdot\left[\frac{1}{(1+1)^{3/2}}+\frac{1}{1}\right]$$

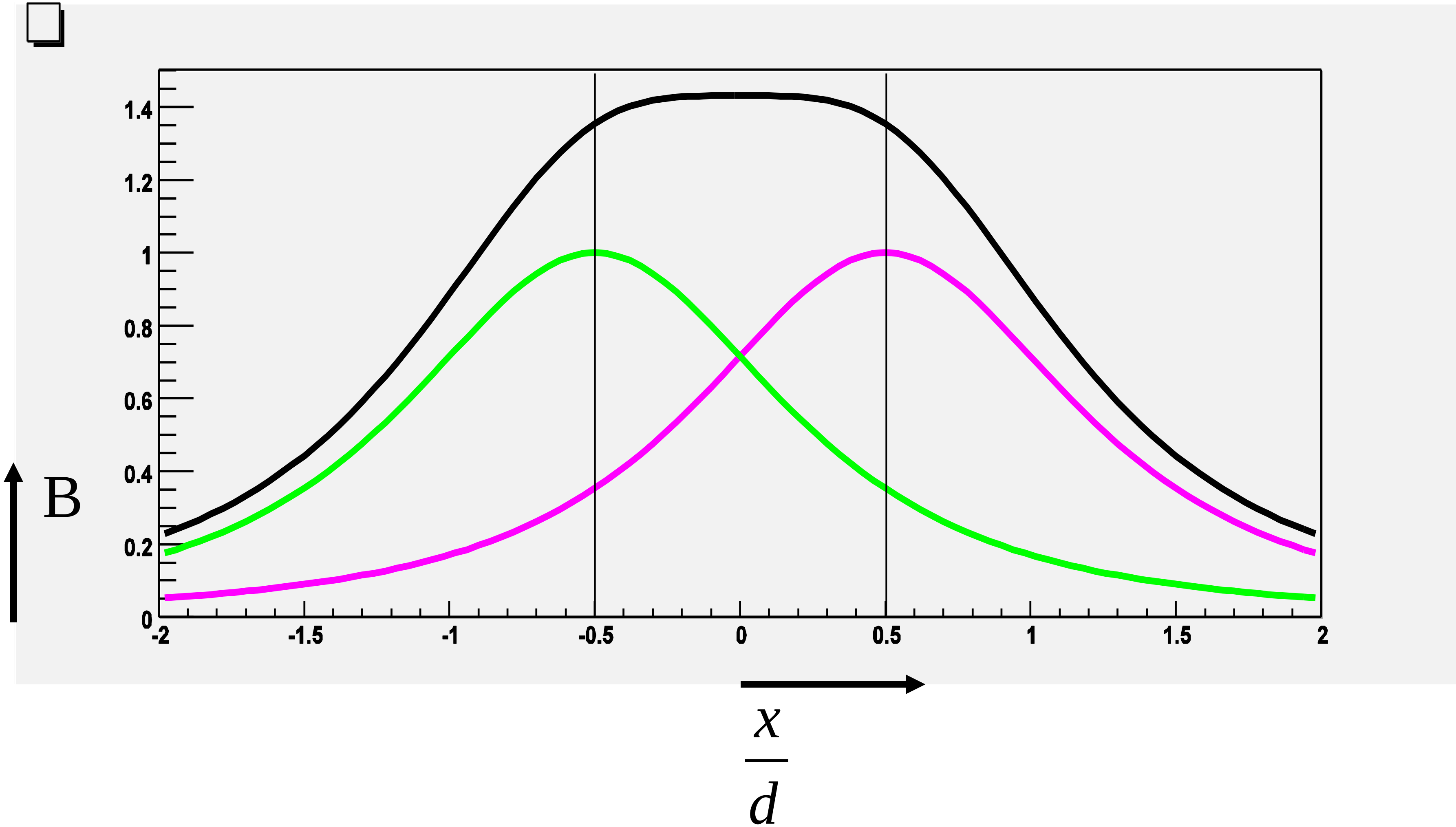
$$B(0)=B_0\cdot 1.43108$$

$$B\left(\frac{d}{2}\right)=B_0\cdot 1.35355$$

$$\underline{\underline{\Delta B/B=5.57\%}}$$

Field of a pair of Helmholtz coils

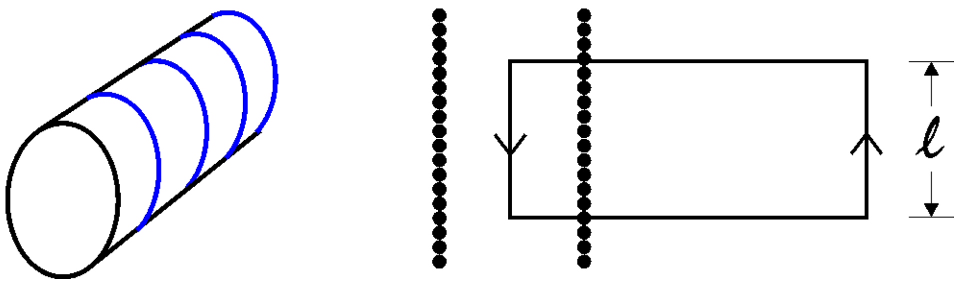
B in units of $\frac{\mu_0 n I}{2a}$



Ampere's law in its integral form: $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$

(I enclosed)

N turns of wire per unit length.
Winding carries a current *I*.
Find B and show it is radially uniform inside the coil.



$B \cdot \ell = \mu_0 \cdot N' \cdot I$ with $N = \dfrac{N'}{\ell}$, thus $\underline{\underline{B = \mu_0 \cdot N \cdot I}}$

Integral independent of path: radially uniform field

Calculate the self-inductance per unit length.

$$L = \frac{\Phi_{tot}}{I} = \frac{B \cdot area}{I} \cdot turns = \frac{\mu_0 NI \cdot \pi R^2}{I} \cdot N\ell = \mu_0 N^2 \pi R^2 \ell$$

... and per length: $\underline{\underline{L/\ell = \mu_0 \pi R^2 N^2}}$

Calculate the magnetic induction and the energy stored.

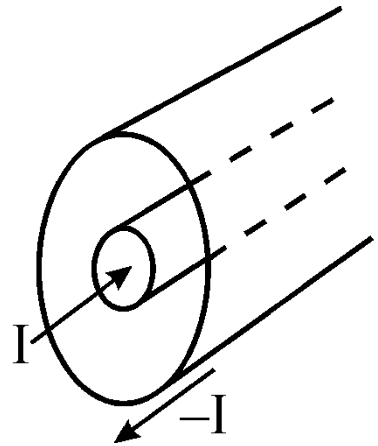
$$R = 0.5\mathrm{m}, \quad \ell = 7\mathrm{m}, \quad N' = 1000 \quad \Rightarrow \quad N = 142.86\mathrm{m}^{-1}$$

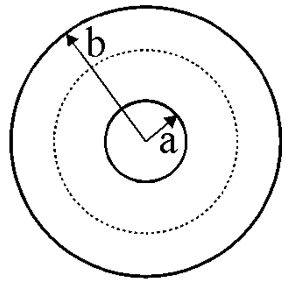
$$\underline{\underline{B}} = \mu_0 NI = 4\pi \cdot 10^{-7} \frac{\mathrm{Vs}}{\mathrm{Am}} \cdot 142.86 \frac{1}{\mathrm{m}} \cdot 5000\mathrm{A} = \underline{\underline{0.897\mathrm{T}}}$$

$$\underline{\underline{U_M}} = \tfrac{1}{2} LI^2 = \tfrac{1}{2} \mu_0 N^2 \pi R^2 \ell \cdot I^2 = \underline{\underline{1.76 \cdot 10^6 \mathrm{J}}}$$

Ampere's Law:

$$\oint \mathbf{B}\cdot d\mathbf{l} = \iint \mathbf{J}\cdot d\mathbf{A} = \mu_0 I$$





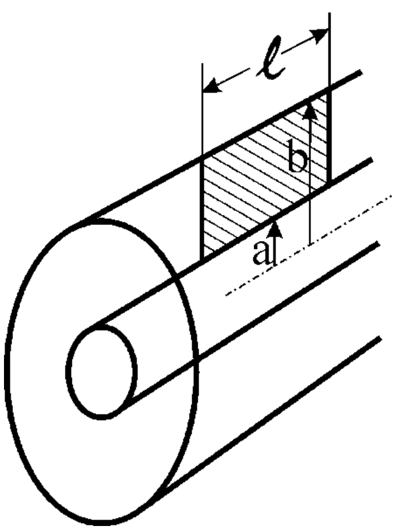
$$b > r > a: \quad 2\pi r B_\theta = \mu_0 I$$

$$r > b: \quad 2\pi r B_\theta = \mu_0 (I - I)$$

$$r < a: \quad 2\pi r B_\theta = 0$$

$$\underline{\underline{B_\theta = \frac{\mu_0 I}{2\pi r}}} \quad \text{only for} \quad b > r > a$$

Calculate the self-inductance:



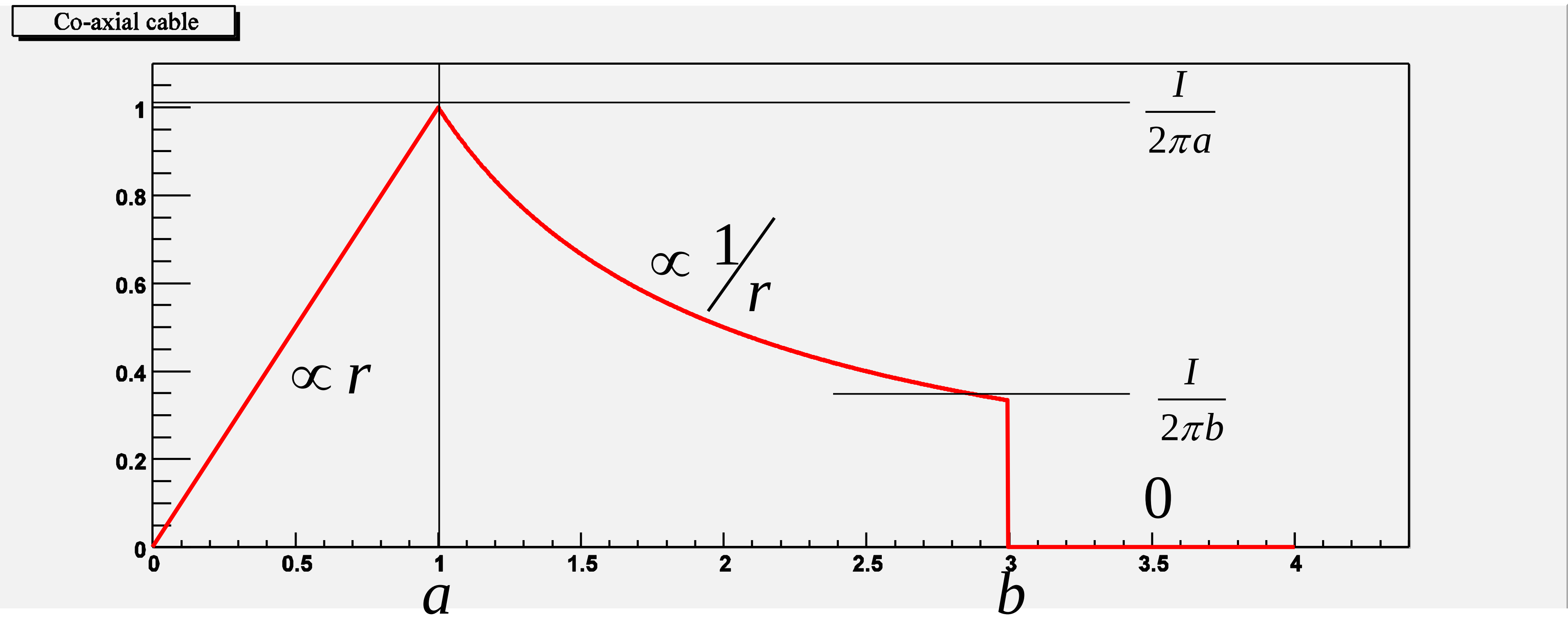
$$\Phi = \int \mathbf{B}\cdot d\mathbf{S} = \int_a^b \frac{\mu_0 I}{2\pi r}\, dr \cdot \ell = \frac{\mu_0 I}{2\pi} \ln\left(\frac{b}{a}\right)\cdot \ell$$

$$L = \frac{\Phi}{I} = \underline{\underline{\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)\cdot \ell}}$$

Sketch the magnitude of B when the inner cylinder is replaced by a solid wire

for $r > a$: see before

for $r < a$: $2\pi r B_\theta = \mu_0 I \cdot \dfrac{\pi r^2}{\pi a^2}$ thus $\underline{\underline{B_\theta = \dfrac{\mu_0 I}{2\pi a} \cdot \dfrac{r}{a}}}$



State the laws of electromagnetic induction

$$emf = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{cut}}{dt}$$

An *emf* will be created such as to counteract a change of current, etc

Faraday disc (thickness *d*).

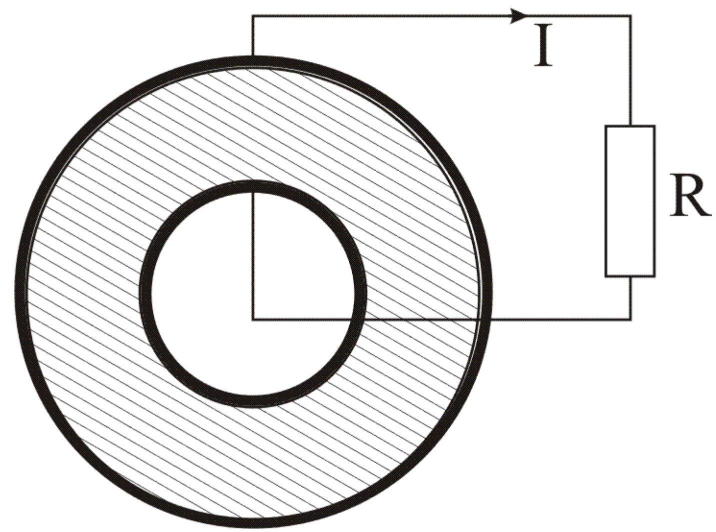
Brushes around entire inner and outer perimeter.

Magnetic flux density along axis of rotation.

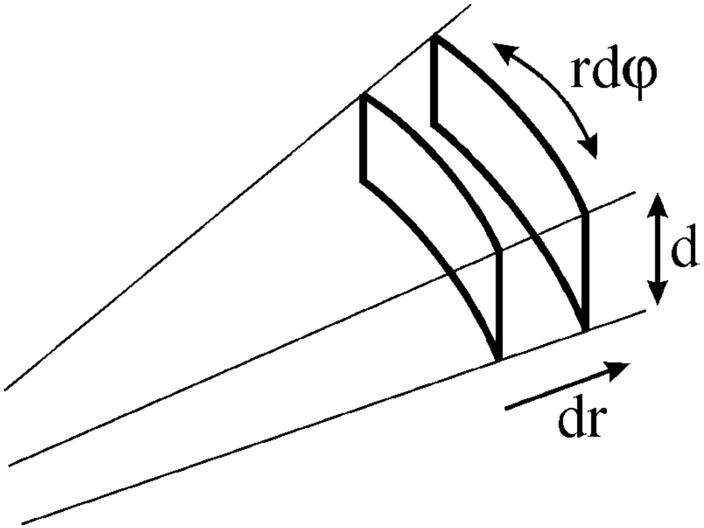
$$r_{inner} = a/2$$

$$r_{outer} = a$$

resistivity ρ



Calculate the electrical resistance



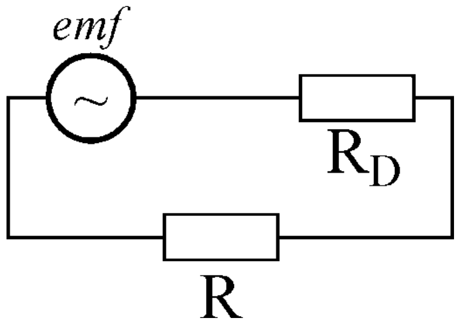
$$R = \rho \cdot \frac{\text{length}}{\text{area}}$$

$$= \int_{a/2}^{a} \rho \cdot \frac{\mathrm{d}r}{2\pi r \cdot d} = \frac{\rho}{2\pi d} \cdot \ln(2)$$

Find the potential difference for the disc rotating in a magnetic flux density B

$$emf = \frac{\text{flux cut}}{\text{time}} = \frac{B \cdot \pi \left(a^2 - \left(\frac{a}{2} \right)^2 \right)}{2\pi / \omega} = \underline{\underline{\frac{3}{8} B a^2 \omega}}$$

Find the optimum value for a load resistor



$$I = \frac{emf}{R_D + R}$$

Power in load:

$$P = \frac{R}{R_D + R} \cdot \frac{(emf)^2}{R_D + R} = (emf)^2 \cdot \frac{R}{(R_D + R)^2}$$

$$\frac{\partial P}{\partial R} = 0: \quad 0 = (R_D + R)^2 \cdot 1 - 2(R_D + R) \cdot R$$

maximum power transfer for: $\underline{\underline{R = R_D}}$

Define magnetic flux and state Faraday's law of electromagnetic induction.

$$\Phi = B \cdot area \qquad \text{and} \qquad emf = -\frac{d\Phi}{dt}$$

Calculate the resistance of the disc R_D measured between the brushes.

$$R = \rho \cdot \frac{\ell}{area} \qquad \text{here: } area(r) = 2\pi r \cdot t$$

$$\underline{\underline{R_D}} = \rho \cdot \int_{a/4}^{a} \frac{dr}{2\pi r t} = \frac{\rho}{2\pi t} \cdot \ln(4) = \underline{\underline{\frac{\rho \ln(2)}{\pi t}}}$$

Find the potential difference between the brushes:

$$\underline{\underline{emf}} = \frac{B \cdot A}{2\pi} \cdot \omega = \frac{\omega B}{2\pi} \cdot \pi a^2 \left(1 - \tfrac{1}{16}\right) = \underline{\underline{\frac{15}{32} \omega B a^2}}$$

A load resistance R_L is connected across the generator and the drive is removed. Calculate τ.

$$E_{rot} = \frac{1}{2} I \omega^2 = \frac{1}{4} m a^2 \omega^2$$

$$\frac{dE_{rot}}{dt} = -P_{dissipated} = -\frac{(emf)^2}{R_D + R_L} = -\left(\frac{15}{32}\right)^2 \cdot \frac{B^2 a^2}{R_D + R_L} \cdot \omega^2$$

and $$\omega^2 = \frac{4E_{rot}}{ma^2}$$

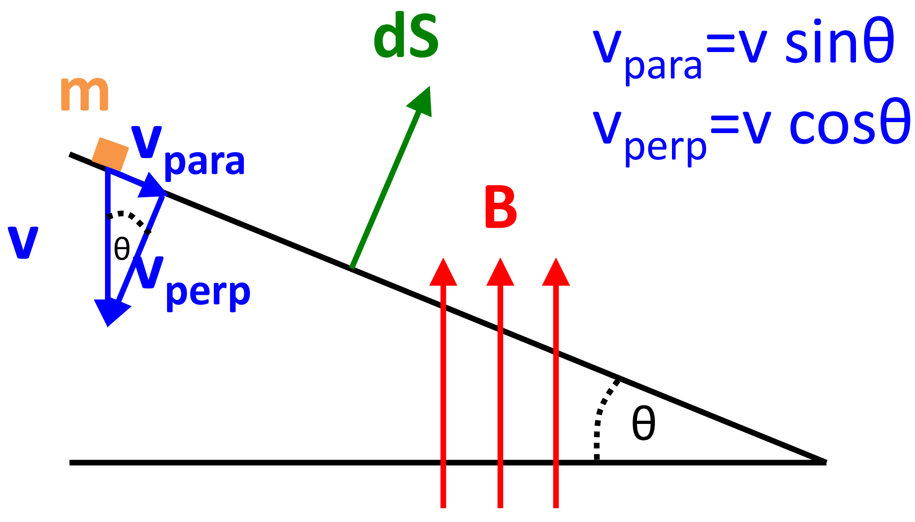
$$\frac{dE_{rot}}{dt} = -\left(\frac{15}{32}\right)^2 \cdot \frac{4B^2a^2}{m\left(R_D + R_L\right)} E_{rot}$$

$$\ln\left(\frac{E_{rot}(t)}{E_{rot}(0)}\right) = -\left(\frac{15}{32}\right)^2 \cdot \frac{4B^2a^2}{m\left(R_D + R_L\right)} \cdot t$$

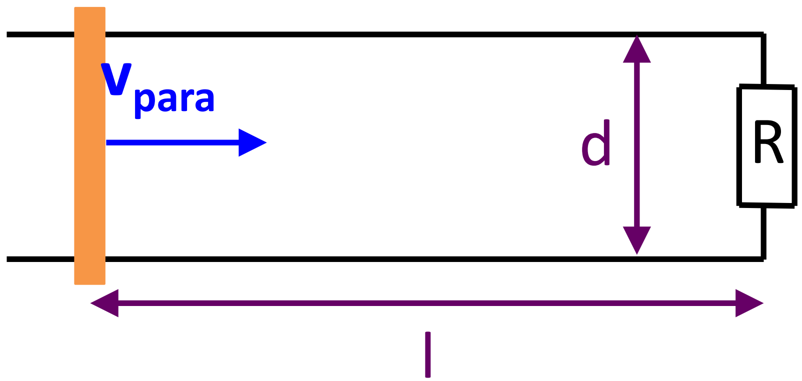
"half its angular speed": $$\frac{E_{rot}(t)}{E_{rot}(0)} = \frac{1}{4}$$

$$\underline{\underline{\tau = \left(\frac{32}{15}\right)^2 \cdot \frac{m\left(R_D + R_L\right)\ln(2)}{2a^2B^2}}}$$

Two parallel rails separated by a distance *d* lie along the direction of greatest slope on an incline making an angle θ with the horizontal. A flat bar of mass *m* rests horizontally across the rails at the top of the incline. Both the bar and the rails are good conductors and the rails are joined by a large resistance *R* at the bottom of the incline. A uniform, vertical magnetic field of flux density **B** exists throughout the region.



side view



top view

Induced e.m.f. $$V_{emf} = -\frac{d}{dt}\int B \bullet dS = B \cos\theta \frac{dA}{dt}$$

where $A = d\, l$

$$V_{emf} = -B\cos\theta\, d\, \frac{dl}{dt} = B \cos\theta\, d\, \mathrm{v}_{para}$$

Induced current: $I = V_{emf}/R$

Equation of Motion - consider magnetic (Lorentz) force on current-carrying wire: $\mathbf{dF} = I\, \mathbf{dl} \times \mathbf{B}$

$$\rightarrow \mathrm{F}_{para} = I\, d\, B \cos\theta = V_{emf}/R\;\; d\, B \cos\theta = B^2 d^2 \cos^2\theta\, /R\, \mathrm{v}_{para}$$

Equation of Motion: $$\mathrm{m} \frac{d}{dt} \mathrm{v}_{para} = \underbrace{mg \sin\theta}_{\text{gravitational}} - \underbrace{B^2 d^2 \cos^2\theta\, /R\;\; \mathrm{v}_{para}}_{\text{magnetic}}$$

$$\rightarrow \frac{d}{dt} \mathrm{v}_{para} + \underbrace{B^2 d^2 \cos^2\theta\, /Rm}_{k}\; \mathrm{v}_{para} = g \sin\theta$$

Solving Equation of Motion: $\frac{d}{dt}\,\mathrm{v}_{para} + k\,\mathrm{v}_{para} = g\,\sin\theta$

try $\mathrm{v}_{para} = A\,\exp(-k\,t) + B \xrightarrow{\text{insert into EoM}} B = \sin\theta\; g/k$

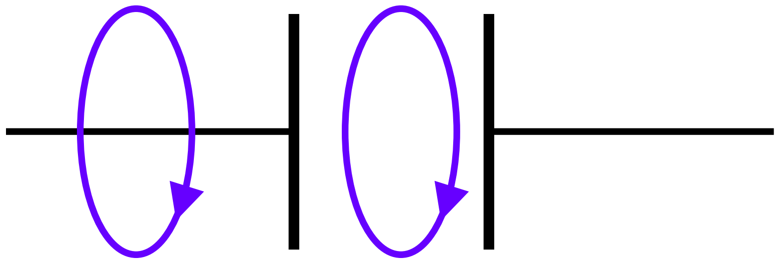
boundary condition: at $t=0$, $\mathrm{v}_{para}=0 \longrightarrow A = -B$

$$\longrightarrow \quad \mathrm{v}_{para} = \sin\theta\; g/k\,(1 - \exp(-\,k\,t))$$

for $t \rightarrow \infty$, constant velocity: $\mathrm{v}_{para,\infty} = \sin\theta\; g/k$

$$\longrightarrow \quad \mathrm{v}_{para,\infty} = g\,m\,R\,\sin\theta \,/\, (B^2\,d^2\,\cos^2\theta)$$

Explain why a displacement current is needed:



$$\int \mathbf{B} \cdot \mathbf{dl} = \mu_0 I$$

$$I_{\mathrm{d}} = \dot{Q} = \dot{\sigma} A = \varepsilon_0 \frac{dE_{\perp}}{dt} A = \varepsilon_0 A \frac{dE}{dt}$$

Show Ampere's Law in differential form:

$$\oint \mathbf{B} \cdot \mathbf{dl} = \mu_0 I_{\mathrm{C}} + \mu_0 I_{\mathrm{D}} = \mu_0 \left[\iint \mathbf{J}_{\mathrm{C}} \mathbf{dA} + \iint \mathbf{J}_{\mathrm{D}} \mathbf{dA} \right]$$

Stokes: $\oint \mathbf{B} \cdot \mathbf{dl} = \iint \nabla \times \mathbf{B}\, \mathbf{dA}$

$$\therefore \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dot{\mathbf{E}}$$

Maxwell's equations in free space:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \dot{\mathbf{E}}$$

Wave equation from these:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \dot{\mathbf{B}} = -\mu_0 \varepsilon_0 \ddot{\mathbf{E}}$$

$$\nabla \times \nabla \times \mathbf{E} = \nabla \underbrace{\left(\nabla \cdot \mathbf{E} \right)}_{=0} - \nabla^2 \mathbf{E}$$

wave equation: $\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \ddot{\mathbf{E}} = 0$

with: $\exp\left[i\left(\omega t \pm kx \right) \right] \ : \ v = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Maxwell's equations with charges and currents:

$$\nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad \nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J}_C + \varepsilon_0 \dot{\mathbf{E}} \right)$$

Wave equation as before (for E and B fields):

$$\Delta \mathbf{E} - \varepsilon_0 \mu_0 \ddot{\mathbf{E}} = 0 \qquad \text{and} \qquad \Delta \mathbf{B} - \varepsilon_0 \mu_0 \ddot{\mathbf{B}} = 0$$

$$\text{from} \quad \exp\left[i \left(\omega t \pm kz \right) \right]: \quad -k^2 + \mu_0 \varepsilon_0 \omega^2 = 0$$

$$\frac{\omega}{k} = \pm \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \pm c \quad \text{(speed of light)}$$

Plane wave solution with E_y and B_x only:

$$\nabla \times \mathbf{E} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & E_y & 0 \end{vmatrix} = \begin{pmatrix} -\dfrac{\partial E_y}{\partial z} \\ 0 \\ 0 \end{pmatrix} = -\dot{\mathbf{B}} = \begin{pmatrix} -\dfrac{\partial B_x}{\partial t} \\ 0 \\ 0 \end{pmatrix}$$

E_y is not a function of x

$$\frac{\mathrm{d}E_y}{\mathrm{d}z} = \frac{\mathrm{d}B_x}{\mathrm{d}t} = \frac{\mathrm{d}B_x}{\mathrm{d}z} \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

integrate over z:

$$E_y = B_x \cdot \frac{\mathrm{d}z}{\mathrm{d}t}$$

Direction of propagation:

$$\frac{\partial}{\partial z}\exp\left[i\left(\omega t\mp kz\right)\right]=\mp ik\exp\left[i\left(\omega t\mp kz\right)\right]$$

$$\frac{\partial}{\partial t}\exp\left[i\left(\omega t\mp kz\right)\right]= \quad i\omega\exp\left[i\left(\omega t\mp kz\right)\right]$$

$$\frac{dz}{dt}= \quad \mp\frac{\omega}{k} \quad = \quad \mp c \quad \text{(in a vacuum)}$$

$$E_{\mathrm{y}} = \quad \mp c\cdot B_{\mathrm{x}}$$

for wave in positive z-direction "–"

for wave in negative z-direction "+"

… or: use Poynting vector:
to give direction of energy flow

$$\mathbf{N}=\frac{1}{\mu_0}\cdot\mathbf{E}\times\mathbf{B}$$